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INTEGRAL MANIFOLD IN SYSTEM DESIGN WITH APPLICATION TO FLEXIBLE LINK ROBOT CONTROL

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Throughout this thesis, results obtained from the manifold approach are shown to be consistent with, and sometimes even extend, some established results in singularly perturbed systems. Keywords: manipulators, optimal control. (KR)

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INTEGRAL MANIFOLD IN SYSTEM DESIGN
WITH APPLICATION TO FLEXIBLE LINK ROBOT CONTROL

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THESIS

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TABLE OF CONTENTS

1. INTRODUCTION

1.1. Definition of an Integral Manifold

The concept of the manifold has been used as first integrals for classical Hamiltonian systems from 1700-1800. In the context of this thesis, as a decomposition tool, the ideas originated from [1-5]. A definition of an integral manifold is now given.

For the following system of differential equations,

$$\dot{x} = f(x, y, t) \tag{1.1.1}$$

$$\dot{y} = g(x, y, t) \tag{1.1.2}$$

where $x \in R^n$, $y \in R^m$ and $t \in R$

a set $M \subset R^n \times R^m \times R$ is said to be an integral manifold for (1.1.1)-(1.1.2) if for $(x_0, y_0, t_0) \in M$, the solution $(x(t), y(t), t)_{|x(t_0)=x_0, y(t_0)=y_0}$ is in M for all t ∈ R.

In other words,

$$y = h(x, t) \tag{1.1.3}$$

is an integral manifold for (1.1.1)-(1.1.2) if given the initial conditions $(x(t_0), y(t_0), t_0)$ that satisfy

$$y(t_0) = h(x(t_0), t_0)$$

we have (1.1.3) hold for all $t \in R$. The flow on this manifold is governed by the n-dimensional system

$$\dot{x} = f(x, h(x, t), t). \tag{1.1.4}$$

Note from (1.1.4) that we are dealing with an n-th order differential equation rather than the (n+m)-th one in the original system (1.1.1)-(1.1.2).

Some of the advantages of using the integral manifolds in systems and control are as follows:

(i) reduction of computational complexity due to system order reduction,

(ii) accounting for the intrinsic slow effect of parasitics in singularly perturbed systems by treating the parasitics state y as $y = h(x, t, \epsilon)$, a function of other state x, perturbation

parameter ϵ , and possibly time variable t.

(iii) an analysis tool to understand some known phenomena, e. g., unsatisfactory performance of flexible robots in high-frequency maneuvers and minimum fuel paths in long-range cruises.

Applications of the integral manifold theory abound in many areas. Some of its applications are flexible joint robot control [6], slow adaptation in adaptive control [7], tracking and disturbance rejection in nonlinear systems [8], power system modeling [9], and synchronous machine modeling [10].

Mathematical treatment of the integral manifold theory, as in [11], sometimes is too restricted in relevant control problems. On the other hand, there are many special features in specific control problems that can be of great use when the manifold approach is being employed. It is this gap that we want to fill in this thesis.

A summary of the research being done in this thesis is as follows.

• **Existence of integral manifolds in the linear system:** an integral manifold in the form of invariant subspace, $z = Lx$, is postulated for the linear system:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = A \begin{bmatrix} x \\ z \end{bmatrix} \tag{1.1.5}$$

Under some assumptions on the A_{ij} entries of the matrix A the existence of such a manifold is guaranteed. Necessary and sufficient conditions for $z = Lx$ to be a linear integral manifold for the linear system are also given in terms of an identity relating L to the A_{ij} entries. An explicit expression for one such L is found and represented by the slow eigenspace of the system matrix A. Once the existence of the linear integral manifold for (1.1.5) is assured, we proved that there also exists a shifted manifold $z = Lx + p(u)$ for the system with input u, i. e.,

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = A \begin{bmatrix} x \\ z \end{bmatrix} + Bu$$

In other words, there exists a family of input dependent shifted manifolds. Singularly per-

turbed systems are treated as a special case, and explicit expressions are found for L and p as asymptotic series in ϵ. Linear time-varying systems are also discussed as an extension to the linear time invariant case with some extra conditions imposed on the $A_{ij}(t)$ entries.

• **Flexible link manipulators:** Modeling flexible link robots leads us to a singularly perturbed system in which the perturbation parameter is related to the reciprocal of the flexibility constant. We interpret the presence of flexibility in the robots as a cause for phase delay in its performance, especially at high frequency maneuvers. Time domain analysis using the manifold approach leads us to a phase delay corrective scheme equivalent to that from the frequency domain analysis. A case study of an interconnected mechanical system, which shares the same basic principle in the modeling of flexible joint robot as in [6], reveals the fact that the overall system has a perturbed natural frequency and a perturbed damping ratio due to the presence of flexibility in the interconnection.

• **Tracking and disturbance rejection in nonlinear systems:** For a class of *linear equivalent nonlinear system with nonlinear output*, we designed a controller that not only linearizes and stabilizes the nonlinear system but also achieves tracking and disturbance rejection of unknown but slowly varying signals.

• **Optimal control systems:** We proved the unique existence of a lower order optimal control problem that is equivalent to a singularly perturbed linear system with a quadratic cost functional to be minimized. The trajectory of the singularly perturbed optimal system is characterized by a fast convergence, with $O(\epsilon)$ cost, to a manifold to which the subsequent motion is restricted. Complete separation of a singularly perturbed optimal system into two is also given. One of these corresponds to the optimal problem as restricted to the manifold, whereas the other one is an optimal problem concerning the convergence of the trajectory to the manifold.

1.2. Existence of Integral Manifolds

(a) We are mainly interested in the singularly perturbed system

$$\dot{x} = f(x, y, t, \epsilon) \tag{1.2.1.a}$$

$$\epsilon \dot{y} = g(x, y, t, \epsilon) \tag{1.2.1.b}$$

where $x \in R^n$, $y \in R^m$, $t \in R$, and ϵ is small positive number. $\dot{x} = \frac{dx}{dt}$ etc.,

Conditions for the existence of an integral manifold, $y = h(x, t, \epsilon)$, for system (1.2.1) are the following[1].

M1: Setting $g(x, y, t, 0) = 0$ gives the isolated solution $y = h^0(x, t)$ for $x \in R^n$, $t \in R$.

M2: Functions f, g and h^0 are all C^2 functions for $x \in R^m$, $|y - h^0(x)| \leq \rho$, $t \in R$, and $0 \leq \epsilon \leq \epsilon_0$, where ρ and ϵ_0 are some positive nonzero numbers.

M3: *All the eigenvalues of* $\frac{\partial g}{\partial y}$ *evaluated at* $(x, h^0(x, t), t, 0)$ *have negative real parts.*

i. e.,

$$\mathrm{Re}\left[\lambda_i\left(\frac{\partial g}{\partial y}\right)\right]_{(x, h^0(x, t), t, 0)} < 0, \quad 1 \leq i \leq n$$

where λ_i's represent eigenvalues.

Comments: M3 is a necessary and sufficient condition for trajectories with initial conditions not on the integral manifold to converge to the manifold asymptotically. It is a local result and is applicable to those trajectories with initial conditions within the region of attraction of the manifold. We will therefore refer to the integral manifolds of the systems satisfying M3 as "attractive." Condition M3 can be relaxed to only requiring that the Jacobian $\frac{\partial g}{\partial y}$ be nonsingular. Detailed proof can be found in [11].

(b) For the periodic nonlinear system

$$\frac{dx}{dt} = X(x, y, t) \tag{1.2.2.a}$$

$$\frac{dy}{dt} = Y(x, y, t) \tag{1.2.2.b}$$

where $x \in R^n$, $y \in R^m$, $t \in R$, and

$$X(x, y, t + \omega) = X(x, y, t)$$

$$Y(x, y, t + \omega) = Y(x, y, t)$$

there exists a bounded periodic integral manifold $y = g(x, t)$ for (1.2.2), i. e., $\|g(x, t)\| \leqslant K$ and $g(x, t + \omega) = g(x, t)$, if

(i) X and Y are continuous and have continuous and bounded partial derivatives with respect to x and y for all x and t, and for $\|y\| \leqslant K$ where $\|.\|$ denotes the Euclidean norm and K is a positive number.

(ii) $\frac{d}{dt}\|y\| < 0$, for all x and t, and $\|y\| = K$, ie $\|y\| \leqslant K$, for all x and t.

(iii) Let

$$V(x, y, t) = \frac{1}{2}\left\{\frac{\partial X}{\partial x} + \left(\frac{\partial X}{\partial x}\right)^T\right\}$$

$$W(x, y, t) = \frac{1}{2}\left\{\frac{\partial Y}{\partial y} + \left(\frac{\partial Y}{\partial y}\right)^T\right\}$$

where $V(x, y, t)$ has eigenvalues $\lambda_k(x, y, t)$, $1 \leqslant k \leqslant n$, and $W(x, y, t)$ has eigenvalues $\mu_j(x, y, t)$, $1 \leqslant j \leqslant m$; $\frac{\partial X}{\partial x}$ denotes the partial derivatives of X with respect to x and A^T denotes the transpose of A.

Also let

$$\lambda = \underset{k}{Min}\{\lambda_k\}, \quad \mu = \underset{j}{Max}\{\mu_j\}$$

λ and μ have the properties that

$$\lambda \geqslant \mu \text{ and } \mu < 0.$$

(iv)

$$4\alpha\beta < (\lambda - \mu)^2$$

where

$$\|\frac{\partial X}{\partial y}\| \leqslant \alpha \ , \ \|\frac{\partial Y}{\partial x}\| \leqslant \beta \ .$$

Proof can be found in [5].

(c) Center manifolds

For the following autonomous system

$$\dot{x} = Ax + f(x, y) \tag{1.2.3.a}$$

$$\dot{y} = By + g(x, y) \tag{1.2.3.b}$$

where $x \in R^n$, $y \in R^m$.

$y = h(x)$ is a center manifold to (1.2.3) if h is smooth, $h(0) = 0$, and $h'(0) = 0$.

Conditions for the existence of center manifolds are the following:

(i) $f \in C^2$, $g \in C^2$ and $f(0, 0) = f'(0, 0) = 0$, $g(0, 0) = g'(0, 0) = 0$.

(ii) $\mathrm{Re}(\lambda_i(A)) < 0$, $1 \leqslant i \leqslant n$, i. e., all the eigenvalues of A have negative real parts.

(iii) $\mathrm{Re}(\lambda_j(B)) = 0$, $1 \leqslant j \leqslant m$.

If (1.2.3) satisfies (i)-(iii), then there exists a center manifold $y = h(x)$ for $|x| < \delta$, $h \in C^2$, and $t \in R$.

Proof is based on the contraction mapping principle. Details can be found in [2].

2. INTEGRAL MANIFOLDS IN LINEAR SYSTEMS

2.1. Introduction

Linear systems are special cases of nonlinear systems. In our investigation on integral manifolds, all the results from nonlinear systems are applicable in linear cases. Due to its linear structure, the application of superposition and Laplace transform are made possible. It is through this that we gain insights into the geometry and analysis of integral manifolds in the control theory. For the ease of illustration, we will concentrate on linear time-invariant systems and treat time-varying systems as an extension of the time-invariant cases.

2.2. Existence of Linear Integral Manifolds in Linear Systems

Start with the following linear time-invariant system:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \tag{2.2.1}$$

where $x \in R^n$, $z \in R^m$, $A_{11} \in R^{n \times n}$, $A_{12} \in R^{n \times m}$, $A_{21} \in R^{m \times n}$, and $A_{22} \in R^{m \times m}$ are constant matrices.

Assumption 2.2.1: A_{22} is nonsingular.

Due to the linearity of (2.2.1), we shall propose a linear integral manifold of the form

$$z = Lx \tag{2.2.2}$$

where $L \in R^{m \times n}$ is a constant matrix.

Lemma 2.2.1

For $z = Lx$ to represent a linear integral manifold for the linear system (2.2.1), it is necessary and sufficient for L to satisfy the following identity.

$$A_{21} + A_{22}L = L(A_{11} + A_{12}L) \tag{2.2.3}$$

Proof:

Differentiate both sides of (2.2.2).

$$\dot{z} = L\dot{x} \tag{2.2.4}$$

Since (x, z) is governed by (2.2.1), we have upon substituting (2.2.1) and (2.2.2) into (2.2.4)

$$(A_{21}+A_{22}L)x = L(A_{11}+A_{12}L)x \ , \tag{2.2.5}$$

which results in the identity (2.2.3) and completes our proof on the necessity part. The sufficiency part follows in the reverse order trivially by noting that the invariant subspace $z = Lx$ is one of the integral manifolds subject to (2.2.1).

QED

The main requirement for the existence of a solution L to (2.2.3) is Assumption 2.2.1. Details can be found in [12].

When our linear system is in singularly perturbed form we have the following result based on a similar argument.

Corollary 2.2.1

The singularly perturbed system

$$\begin{bmatrix} \dot{x} \\ \epsilon\dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = A \begin{bmatrix} x \\ z \end{bmatrix} \tag{2.2.6}$$

where $\epsilon \in [-\epsilon^*, \epsilon^*]$, $\epsilon^* > 0$ is a small positive number, has a linear integral manifold $z = Lx$ and L satisfies

$$A_{21} + A_{22}L = \epsilon L(A_{11} + A_{12}L) \tag{2.2.7}$$

Moreover, L can be solved as

$$L = -A_{22}^{-1}A_{21} + O(\epsilon) \tag{2.2.8}$$

Proof:

Solution of L can be found by equating coefficients of different powers of ϵ on both sides of (2.2.7).

QED

Note with assumption I our singularly perturbed system (2.2.6) exhibits a two-time scale property due to a clear separation of eigenvalues into two groups, small and large, respec-

tively. To facilitate our discussion on the more general system (2.2.1), we adopt the following assumption.

Assumption 2.2.2:

System (2.2.1) possesses n relatively small and m relatively large eigenvalues .

In this context, without loss of generality, we shall refer to x as the "slow" mode and z as the "fast" mode in the subsequent discussion. The existence of a solution to (2.2.3) is guaranteed through Assumption 2.2.1, and some bounds on the A_{ij} entries which in turns are related to Assumption 2.2.2[12]. We thus have a linear integral manifold for system (2.2.1). When system (2.2.1) is restricted to the invariant subspace characterized by $z = Lx$, the slow variable is governed by

$$\dot{x} = (A_{11} + A_{12}L)x = A_s x \tag{2.2.9}$$

We now state a fact on how L is expressed in terms of the slow eigenspace of (2.2.1).

Lemma 2.2.2

If $v_s = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is a slow eigenspace for (2.2.1) and v_1 is nonsingular, where $v_1 \in R^{n \times n}$, $v_2 \in R^{m \times n}$, then $L = v_2 v_1^{-1}$. is one of the solutions to (2.2.3) subject to the linear system (2.2.1). Moreover L is independent of the basis chosen for the slow eigenspace v_s .

Proof:

We will show that (2.2.3) is equivalent to the following:

$$\begin{bmatrix} L & -I_m \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I_n \\ L \end{bmatrix} = 0 \tag{2.2.10}$$

where $I_n \in R^{n \times n}$, $I_m \in R^{m \times m}$ are identity matrices.

Expand (2.2.10) by multiplying out the matrices and using (2.2.3), we get

$$L(A_{11} + A_{12}L) - (A_{21} + A_{22}L) = 0 ,$$

thus verifying (2.2.10).

Suppose $v_s = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is a slow eigenspace for (2.2.1). Multiply v_1 on both sides of (2.2.10) and

take $L = v_2 v_1^{-1}$

$$\begin{bmatrix} L & -I_m \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} I_n \\ L \end{bmatrix} v_1 = \begin{bmatrix} v_2 v_1^{-1} & -I_m \end{bmatrix} \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} . \tag{2.2.11}$$

Since v_s is a slow eigenspace of (2.2.1), we have

$$A \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Lambda_s \tag{2.2.12}$$

where Λ_s is a diagonal matrix which contains the n *small* eigenvalues of A as its diagonal elements. In a more general sense, Λ_s can be a matrix in Jordan canonical form.

Hence, (2.2.11) becomes

$$\begin{bmatrix} L & -I_m \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Lambda_s = \begin{bmatrix} v_2 v_1^{-1} & -I_m \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \Lambda_s = (v_2 v_1^{-1} v_1 - v_2) \Lambda_s = 0 .$$

This proves that $L = v_2 v_1^{-1}$ is one of the solutions. To see that L is independent of the basis of v_s, we take a new basis

$$v'_s = v_s M = \begin{bmatrix} v_1 M \\ v_2 M \end{bmatrix} = \begin{bmatrix} v'_1 \\ v'_2 \end{bmatrix}$$

where $M \in R^{n \times n}$ and is nonsingular.

$$v'_2 v'^{-1}_1 = (v_2 M)(M^{-1} v_1^{-1}) = v_2 v_1^{-1}$$

This completes our proof.

QED

For linear time-varying systems we consider

$$\begin{bmatrix} \dot{x} \\ \epsilon \dot{z} \end{bmatrix} = A(t) \begin{bmatrix} x \\ z \end{bmatrix} \tag{2.2.13}$$

where ϵ is a small number, and A is the same dimension as in (2.2.1).

Assumption 2.2.3: In our domain of interest D, $A_{ij}(t)$ are continuously differentiable and bounded, and $\dot{A}_{12}$, $\dot{A}_{21}$ and $\dot{A}_{22}$ are bounded.

With this assumption we are also assured of the existence of a linear integral manifold z =L(t) x, where L(t) satisfies

$$\epsilon \dot{L} = A_{21} + A_{22}L - \epsilon L(A_{11} + A_{12}L) \tag{2.2.14}$$

The solvability of L in (2.2.14) is guaranteed by Assumption 2.2.3. Detailed proof can be found on p. 212 of [13].

Once the existence of the integral manifold for our linear system is assured, the question regarding the existence of the integral manifold for the same system with input is best answered by the next theorem.

Theorem 2.2.1

If a linear system without input possesses a linear integral manifold M_0, characterized by z=Lx, then for every piecewise continuous and Laplace transformable input u to the same system it has a linear integral manifold M_u characterized by $z = Lx + p$ and p satisfies

$$\dot{p} = (A_{22} - LA_{12})p + (B_2 - LB_1)u \,. \tag{2.2.15}$$

Proof:

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = A(t)\begin{bmatrix} x \\ z \end{bmatrix} \tag{2.2.16.a}$$

Solution to (2.2.16.a) is given by

$$\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \Phi(t\,,t_0)\begin{bmatrix} x(t_0) \\ z(t_0) \end{bmatrix} \tag{2.2.16.b}$$

where $\Phi(t\,,t_0)$ is the state transition matrix and satisfies

$$\dot{\Phi} = A(t)\Phi.$$

When the linear system (2.2.16.a) has an input, we write

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = A(t)\begin{bmatrix} x \\ z \end{bmatrix} + B(t)u(t)\,. \tag{2.2.17}$$

Recall that $\Phi(t\,,t_0)$ is the state transition matrix of (2.2.16.a), the complete solution to (2.2.16.b) is

$$\begin{bmatrix} x(t) \\ z(t) \end{bmatrix} = \Phi(t, t_0)\begin{bmatrix} x(t_0) \\ z(t_0) \end{bmatrix} + \int_{t_0}^{t} \Phi(t, \tau)B(\tau)u(\tau)d\tau$$

$$= \Phi(t, t_0)\begin{bmatrix} x(t_0) \\ z(t_0) \end{bmatrix} + \begin{bmatrix} x_u(t) \\ z_u(t) \end{bmatrix}.$$

Define

$$\begin{bmatrix} \hat{x} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} x \\ z \end{bmatrix} - \begin{bmatrix} x_u \\ z_u \end{bmatrix}. \qquad (2.2.18)$$

Since $\begin{bmatrix} x_u \\ z_u \end{bmatrix}_{|t = t_0} = 0$, we have

$$\begin{bmatrix} \hat{x}(t) \\ \hat{z}(t) \end{bmatrix} = \Phi(t, t_0)\begin{bmatrix} \hat{x}(t_0) \\ \hat{z}(t_0) \end{bmatrix}$$

thus,

$$\begin{bmatrix} \dot{\hat{x}} \\ \dot{\hat{z}} \end{bmatrix} = A(t)\begin{bmatrix} \hat{x} \\ \hat{z} \end{bmatrix} \qquad (2.2.19)$$

If (2.2.16.a) has a linear integral manifold $z = L(t)x$, then (2.2.19) also has one given by $\hat{z} = L(t)\hat{x}$. From (2.2.18)

$$z = \hat{z} + z_u = L\hat{x} + z_u = z_u + L(x - x_u)$$

$$= Lx + (z_u - Lx_u) = Lx + p \qquad (2.2.20)$$

Note that $p(t_0) = 0$.

To see what p should satisfy, we consider

$$z = Lx + p$$

Differentiate both sides,

$$\dot{z} = \dot{L}x + L\dot{x} + \dot{p}$$

Substitute (2.2.20) into the above equation,

$$(A_{21} + A_{22}L)x + A_{22}p + B_2u = \dot{L}x + L[(A_{11} + A_{12}L)x + A_{12}p + B_1u] + \dot{p} \qquad (2.2.21)$$

By the identity on L, i. e., (2.2.3), (2.2.21) simplifies to

$$\dot{p} = (A_{22} - LA_{12})p + (B_2 - LB_1)u \quad . \quad p(t_0) = 0 .$$

QED

Corollary 2.2.2

The singularly perturbed system

$$\begin{bmatrix} \dot{x} \\ \epsilon \dot{z} \end{bmatrix} = A(t) \begin{bmatrix} x \\ z \end{bmatrix} + B(t)u \tag{2.2.22}$$

has a linear integral manifold $z = Lx + p$, where p satisfies

$$\epsilon \dot{p} = (A_{22} - \epsilon LA_{12})p + (B_2 - \epsilon LB_1)u \tag{2.2.23}$$

Furthermore, a steady state solution of p to (2.2.23) can be solved algebraically to any order of ϵ provided the input u does not contain any frequency that is of an order higher than $1/\epsilon$. In fact,

$$p = -A_{22}^{-1}B_2 u + O(\epsilon)$$

Proof:

To solve for the steady state solution for p, we treat both sides (2.2.23) as an asymptotic series of ϵ and use MAE (Matched Asymptotic Expansion).

$$p = p_0 + \epsilon p_1 + \cdots \tag{2.2.24}$$

$$L = L_0 + \epsilon L_1 + \cdots \tag{2.2.25}$$

$$u = u_0 + \epsilon u_1 + \cdots \tag{2.2.26}$$

Collecting terms of ϵ^0 on both sides of (2.2.23),

$$0 = A_{22}p_0 + B_2 u_0$$

Or

$$p_0 = -A_{22}^{-1}B_2 u_0 = -A_{22}^{-1}B_2 u + O(\epsilon) . \tag{2.2.27}$$

thus proving

$$p = -A_{22}^{-1}B_2 u + O(\epsilon) .$$

Similarly for ϵ^1

$$\dot{p}_0 = A_{22}p_1 - L_0 A_{12}p_0 + B_2 u_1 - L_0 B_1 u_0$$

Thus,

$$p_1 = A_{22}^{-1}(\dot{p}_0 + L_0 A_{12} p_0 - B_2 u_1 + L_0 B_1 u_0)$$
$$= A_{22}^{-1}[-A_{22}^{-1} B_2 \dot{u}_0 + L_0(B_1 - A_{12} A_{22}^{-1} B_2) u_0 - B_2 u_1]$$

With the assumption that the input does not contain a frequency as high as $1/\epsilon$, i. e., $|\epsilon \dot{u}_0| < 1$, MAE is valid and we can continue to look for the steady state solution of p up to any order of ϵ in this manner.

QED

2.3. Geometry of Integral Manifolds and Its Relationship to Inputs

The subspace $z = Lx$ is an integral manifold for the linear system (2.2.1) if the solution $(x(t_0), Lx(t_0))$ lies on the subspace z=Lx for all t $\in R$. This defines a clear picture as shown in Figure 2-1. Each solution of the system shall remain in this invariant subspace provided it starts with its initial condition on the manifold. When an input is applied to (2.2.1), the manifold changes to $z = Lx + p$, where p is related to the input u through the differential equation (2.2.15). An input usually consists of feedback, ie closed-loop control, and/or open-loop control. As will be seen later slow manifolds are invariant towards fast feedback. So without loss of generality, we shall consider feedback of slow modes only. Here slow modes are understood to be the state x.

When a feedback of slow modes is applied to (2.2.1) the overall closed-loop system is again another linear system similar to (2.2.1). Hence the resultant closed-loop system has a linear integral manifold for itself. The feedback has effectively shifted the original linear integral manifold to another linear integral manifold.

Lemma 2.3.1

If the linear system (2.2.1) has a linear integral manifold z=Lx, then the resultant system with input u=Kx will have a linear manifold $z = (L + M)x$, where M is related to L and K through

$$\tilde{B}K + \tilde{A}M = M(A_s + B_1 K + A_{12} M) \qquad (2.3.1)$$

where

$$\tilde{A} = A_{22} - LA_{12} \, . \, \tilde{B} = B_2 - LB_1 \, ,$$

$$\text{and } A_s = A_{11} + A_{12}L \, .$$

Proof:

First of all we note that the existence of an integral manifold for the resultant system with input is assured by Theorem 2.2.4. When an input is applied to (2.2.1), it is in the form

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \, . \tag{2.3.2}$$

This can be transformed by Theorem 2.2.4 to

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A_s & A_{12} \\ 0 & \tilde{A} \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} B_1 \\ \tilde{B} \end{bmatrix} u \tag{2.3.3}$$

with

$$z = Lx + p \, . \tag{2.3.4}$$

When u=Kx, (2.3.3) becomes a closed-loop system,

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A_s + B_1K & A_{12} \\ \tilde{B}K & \tilde{A} \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} . \tag{2.3.5}$$

Equation (2.3.5) is a linear system similar to (2.2.1) and has a manifold given by

$$p = Mx \tag{2.3.6}$$

where M satisfies an equation similar to that on L in (2.2.3), i. e.,

$$\tilde{B}K + \tilde{A}M = M(A_s + B_1K + A_{12}M) \, .$$

QED

Note that M is directly related to the feedback gain K as seen from (2.3.1). The overall system with input u =Kx has a shifted manifold $z = (L + M)x$ as shown in Figure 2-2. When we have open-loop control as the input to our system (2.2.1), the resultant system has a time-varying shifted manifold as described by the next Lemma.

Lemma 2.3.2

If the linear system (2.2.1) has a linear integral manifold z=Lx, then the resultant system with a continuous input u=f(t) has an integral manifold, z=Lx+q , where L is as described by

(2.2.3) and q is related to f(t) through

$$\dot{q} = \tilde{A}q + \tilde{B}f(t) \tag{2.3.7}$$

where

$$\tilde{A} = A_{22} - LA_{12},$$

$$\text{and } \tilde{B} = B_2 - LB_1.$$

Proof:

Again the existence of an integral manifold for our linear system with open-loop control is guaranteed by Theorem 2.2.4.

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} f(t). \tag{2.3.8}$$

By Theorem 2.2.4 (2.3.8) has an integral manifold

$$z = Lx + q$$

where q is related to u through an equation of the form (2.2.15),

$$\dot{q} = \tilde{A}q + \tilde{B}f(t).$$

QED

When we have both open-loop and closed-loop control as our input to the linear system (2.2.1), we can use the superposition principle for our system and deduce the following result.

Theorem 2.3.1

If the linear system (2.2.1) has a linear integral manifold $z = Lx$, then there exists a time-varying shifted manifold $z = (L + M)x + q$ for the closed-loop system with input $u = f(t) + Kx$ where M and q satisfy (2.3.1) and (2.3.9), respectively.

Also

$$\dot{q} = \bar{A}q + \bar{B}f(t) \tag{2.3.9}$$

where

$$\bar{A} = A_{22} - (L + M)A_{12},$$

$$\text{and } \bar{B} = B_2 - (L + M)B_1.$$

Proof:

It can be easily shown by Lemma 2.3.1, Lemma 2.3.2 and the principle of superposition.

QED

For linear time-varying systems, we have the same form of manifolds except that L and M are functions of time. When the system does not start with its initial condition on the manifold, there is a deviation from the integral manifold. To investigate this situation we perform an exact transformation on the linear system by using x, the slow mode and η, the deviation from the manifold as the new state space.

Theorem 2.3.2

The linear system

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{2.3.10}$$

is equivalent to

$$\begin{bmatrix} \dot{x} \\ \dot{p} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A_s & A_{12} & A_{12} \\ 0 & \tilde{A} & 0 \\ 0 & 0 & \tilde{A} \end{bmatrix} \begin{bmatrix} x \\ p \\ \eta \end{bmatrix} + \begin{bmatrix} B_1 \\ \tilde{B} \\ 0 \end{bmatrix} u \tag{2.3.11}$$

where

$$\eta = z - Lx - p \tag{2.3.12}$$

is the deviation from the integral manifold of (2.3.10), $z = Lx + p$. L satisfies the equation (2.2.3). A_s , $\tilde{A}$ and $\tilde{B}$ are as defined in Lemma 2.3.2.

Proof:

Substitute (2.3.12) into (2.3.10) results in

$$\dot{x} = A_s x + A_{12} p + A_{12} \eta + B_1 u \ . \tag{2.3.13}$$

Differentiating both sides of (2.3.12)

$$\dot{\eta} = \dot{z} - L\dot{x} - \dot{p}$$

$$= A_{21}x + A_{22}(\eta + Lx + p) + B_2 u$$

$$- L[A_{11}x + A_{12}(\eta + Lx + p) + B_1 u] - \dot{p} .$$

With (2.2.3) it simplifies to

$$\dot{\eta} = (A_{22} - LA_{12})\eta + (A_{22} - LA_{12})p + (B_2 - LB_1)u - \dot{p} .$$

Take

$$\dot{p} = (A_{22} - LA_{12})p + (B_2 - LB_1)u , \tag{2.3.14}$$

we have

$$\dot{\eta} = (A_{22} - LA_{12})\eta . \tag{2.3.15}$$

Combining (2.3.13)-(2.3.15), we have (2.3.11).

QED

We see from (2.3.11) that the differential equation governing η, the deviation from the manifold, is totally decoupled from the rest of the system. So if we assume that $\tilde{A}$ is Hurwitz, then η goes to zero asymptotically and our system will be on the invariant subspace M_u . It is in this context that we refer to the manifold M_u as an attractive manifold.

Corollary 2.3.1

If $\tilde{A}$ is Hurwitz then the steady state of (2.3.11) as restricted to the manifold is equivalent to the following system:

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A_s & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} B_1 \\ \tilde{B} \end{bmatrix} u . \tag{2.3.16}$$

For singularly perturbed systems, we have a similar result.

Corollary 2.3.2

The singularly perturbed system

$$\begin{bmatrix} \dot{x} \\ \epsilon\dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{2.3.17}$$

is equivalent to

$$\begin{bmatrix} \dot{x} \\ \epsilon\dot{p} \\ \epsilon\dot{\eta} \end{bmatrix} = \begin{bmatrix} A_s & A_{12} & A_{12} \\ 0 & \tilde{A} & 0 \\ 0 & 0 & \tilde{A} \end{bmatrix} \begin{bmatrix} x \\ p \\ \eta \end{bmatrix} + \begin{bmatrix} B_1 \\ \tilde{B} \\ 0 \end{bmatrix} u \tag{2.3.18}$$

where η is the fast deviation from the slow manifold $\eta = z - Lx - p$ and L satisfies

$$A_{21} + A_{22}L = \epsilon L (A_{11} + A_{12}L) \tag{2.3.19}$$

and

$$\tilde{A} = A_{22} - \epsilon L A_{12} \, . \, \tilde{B} = B_2 - \epsilon L B_1 \, .$$

$$\text{and } A_s = A_{11} + A_{12}L \, .$$

Furthermore, if A_{22} is Hurwitz, for $\epsilon \in [0, \epsilon^{*}]$, $0 < \epsilon^{*} << 1$, (2.3.18) is equivalent to

$$\begin{bmatrix} \dot{x} \\ \epsilon\dot{p} \end{bmatrix} = \begin{bmatrix} A_s & A_{12} \\ 0 & \tilde{A} \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} B_1 \\ \tilde{B} \end{bmatrix} u \tag{2.3.20}$$

It is observed from (2.3.18) that η is a decoupled fast subsystem. We shall hereafter refer to it as the fast variable. Now we show that the integral manifold of the linear system (2.3.2) is invariant with respect to the feedback of fast variable η.

Theorem 2.3.3

Slow manifolds are invariant towards fast feedback.

Proof:

Any input to our system must be of the form

$$u = Kx + f(t) + G\eta$$

for some constant vector K, G and some continuous function f(t). We shall show that the linear system with such an input,

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \, . \tag{2.3.21}$$

has an integral manifold $z = (L + M)x + q$ regardless of the choices of G, thus proving its invariance with respect to the fast feedback. The variables L, M, and q satisfy (2.2.3),(2.3.1) and (2.3.9), respectively. Due to its linearity we can use the superposition principle to study

the overall effect of different inputs to the system (2.3.21). When the slow feedback control $u_s = Kx$ is applied to (2.3.21), the resulting system becomes

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \bar{A} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A_{11}+B_1K & A_{12} \\ A_{21}+B_2K & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \tag{2.3.22}$$

and has a linear integral manifold given by $z = (L + M)x = \bar{L}x$, where M satisfies (2.3.1). Or we can say that $\bar{L}$ satisfies

$$\bar{A}_{21} + \bar{A}_{22}\bar{L} = \bar{L}(\bar{A}_{11} + \bar{A}_{12}\bar{L}). \tag{2.3.23}$$

When in addition the fast feedback

$$u_f = G\eta = G(z - \bar{L}x - q)$$

$$= G \begin{bmatrix} -L & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} - Gq$$

is also applied, we have from (2.3.22),

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \bar{A} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} b \\ B_2 \end{bmatrix} G \begin{bmatrix} -\bar{L} & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} (-Gq)$$

$$= \begin{bmatrix} \bar{A}_{11} - B_1G\bar{L} & \bar{A}_{12} + B_1G \\ \bar{A}_{21} - B_2G\bar{L} & \bar{A}_{22} + B_2G \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} (-Gq)$$

$$= A^G \begin{bmatrix} x \\ z \end{bmatrix} + B(-Gq). \tag{2.3.24}$$

We now show that

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = A^G \begin{bmatrix} x \\ z \end{bmatrix}$$

has the same linear integral manifold $z = \bar{L}x$ as in (2.3.22).

$$A_{21}^G + A_{22}^G\bar{L} - \bar{L}(A_{11}^G + A_{12}^G\bar{L})$$

$$= \bar{A}_{21} - B_2G\bar{L} + (\bar{A}_{22} + B_2G)\bar{L} - \bar{L}[\bar{A}_{11} - B_1G\bar{L} + (\bar{A}_{12} + B_1G)\bar{L}]$$

$$= \bar{A}_{21} + \bar{A}_{22}\bar{L} - \bar{L}(\bar{A}_{11} + \bar{A}_{12}\bar{L}) = 0$$

by (2.3.23).

This completes our proof for the case when the input is of the form of a closed-loop feedback control, $u = Kx + G\eta$. Now for the system with input $u = Kx + G\eta + f(t)$,

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = A \begin{bmatrix} x \\ z \end{bmatrix} + B(Kx + G\eta + f), \tag{2.3.25}$$

it is equivalent to

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = A^{G} \begin{bmatrix} x \\ z \end{bmatrix} + B(f - Gq) \tag{2.3.26}$$

by using (2.3.24) and the superposition principle. Finally we show that (2.3.26) has the same manifold $z = \bar{L}x + q$ as (2.3.25) does.

By using Lemma 2.3.3 , (2.3.26) has an integral manifold $z = \bar{L}x + q_G$, where q_G satisfies the following:

$$\begin{aligned} \dot{q}_G &= (A^G_{22} - \bar{L}A^G_{12})q_G + (B_2 - \bar{L}B_1)(f - Gq) \\ &= (\overline{A}_{22} - \bar{L}\overline{A}_{12})q_G + (B_2 - \bar{L}B_1)Gq + (B_2 - \bar{L}B_1)(f - Gq) \\ &= (\overline{A}_{22} - \bar{L}\overline{A}_{12})q_G + (B_2 - \bar{L}B_1)f \end{aligned} \tag{2.3.27}$$

By comparing (2.3.27) with (2.3.7), it is obvious that a solution to (2.3.27) is given by $q_G = q$, which is independent of the fast feedback gain G. Hence, (2.3.26) or (2.3.25) has the integral manifold $z = \bar{L}x + q$ that is independent of the fast feedback.

QED

The above theorem enables us to carry out the two-stage design. We can first stabilize our fast subsystem and then concentrate on the slow subsystem by regarding it as subsequently decoupled from the fast subsystem. On the decoupled slow subsystem, we can design our controller to achieve specific tasks, e. g., tracking etc. The eigenvalue placement problem can be done in two steps. Desired fast eigenvalues can be obtained through fast feedback on the fast subsystem. We then work on the slow subsystem to achieve our slow eigenvalue assignment objective. When the system does not start on the manifold, and the fast subsystem is not stable, this is equivalent to saying that the slow manifold is repulsive and the solution will not come down to the slow manifold. We can stabilize our fast subsystem as shown

below.

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = A \begin{bmatrix} x \\ z \end{bmatrix} + B\,(u_s + u_f\,) \tag{2.3.28}$$

is equivalent to

$$\begin{bmatrix} \dot{x} \\ \dot{p} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A_s & A_{12} & A_{12} \\ 0 & \tilde{A} & 0 \\ 0 & 0 & \tilde{A} \end{bmatrix} \begin{bmatrix} x \\ p \\ \eta \end{bmatrix} + \begin{bmatrix} B_1(u_s + u_f\,) \\ \tilde{B}u_s \\ \tilde{B}u_f \end{bmatrix} \tag{2.3.29}$$

where A_s , $\tilde{A}$ and $\tilde{B}$ are as defined in Lemma 2.3.2. Take $u_f = G\eta$ so that $\tilde{A} + \tilde{B}G$ is Hurwitz: (2.3.29) becomes

$$\begin{bmatrix} \dot{x} \\ \dot{p} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A_s & A_{12} & A_{12} + B_1G \\ 0 & \tilde{A} & \tilde{B}G \\ 0 & 0 & \tilde{A} + \tilde{B}G \end{bmatrix} \begin{bmatrix} x \\ p \\ \eta \end{bmatrix} + \begin{bmatrix} B_1 \\ \tilde{B} \\ 0 \end{bmatrix} u_s\,. \tag{2.3.30}$$

When the closed-loop fast subsystem is asymptotically stable, the steady state of the slow subsystem becomes the same as the one restricted to the manifold, i. e.,

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} A_s & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} B_1 \\ \tilde{B} \end{bmatrix} u_s \tag{2.3.31}$$

Thus all the previous results based on systems that start on the manifold can also be applied to systems that do not have their initial conditions on the manifold but have subsystems that are stable or can somehow be stabilized through fast feedback.

2.4. Frequency Domain Interpretation of Integral Manifolds

When our system starts on the manifold, it will "flow" along the manifold as time goes on. The motion on the manifold is governed by a system of differential equations that is of a lesser order than the original system. It is crucial to have the initial condition on the manifold so that we can consider our system as restricted to this invariant subspace as time progresses. For the case when the initial condition is not on the manifold, a separate discussion is also

given in Section 2.3. For now, we shall assume that the initial condition is on the manifold. We then investigate the equivalence of the two designs from the point of view of frequency domain and that of the integral manifolds.

• **Equivalence of the Two Designs**

Theorem 2.4.1

The integral manifold design is equivalent to the frequency domain analysis.

Proof:

When we take the Laplace transform on both sides of the linear system

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \, , \tag{2.4.1}$$

we have for z

$$Z = (s\tilde{I} - A_{22})^{-1} (A_{21}X + B_2U + z(0)) \tag{2.4.2}$$

where $\tilde{I} \in R^{m \times m}$ is an identity matrix.

When (2.4.1) is on the manifold

$$z = Lx + p \, , \tag{2.4.3}$$

we have

$$\dot{x} = (A_{11} + A_{12}L)x + A_{12}p + B_1u \tag{2.4.4}$$

$$\dot{p} = (A_{22} - LA_{12})p + (B_2 - LB_1)u \tag{2.4.5}$$

$$z(0) = Lx(0) + p(0) \, . \tag{2.4.6}$$

We now show that the integral manifold in the frequency domain is the same as (2.4.2).

Rewrite (2.4.5) as

$$\dot{p} = A_{22}p + B_2u - L(A_{12}p + B_1u) \, . \tag{2.4.7}$$

Taking the Laplace transform on (2.4.7), we have

$$P = (s\tilde{I} - A_{22})^{-1}[B_2U - L(A_{12}P + B_1U) + p(0)] \tag{2.4.8}$$

Now we take the Laplace transform on (2.4.4)

$$(sI - A_{11} - A_{12}L)X - x(0) = A_{12}P + B_1U \ . \tag{2.4.9}$$

Multiplying both sides by L , we have

$$[sL - L(A_{11} + A_{12}L)]X - Lx(0) = L(A_{12}P + B_1U) \ . \tag{2.4.10}$$

Using the identity that L satisfies, i. e.,

$$L(A_{11} + A_{12}L) = A_{21} + A_{22}L \ ,$$

(2.4.10) becomes

$$[(s\tilde{I} - A_{22})L - A_{21}]X - Lx(0) = L(A_{12}P + B_1U) \ . \tag{2.4.11}$$

Substituting (2.4.11) into (2.4.8) we have

$$\begin{aligned} P &= (s\tilde{I} - A_{22})^{-1}B_2U - [(s\tilde{I} - A_{22})L - A_{21}]X + Lx(0) + p(0) \\ &= (s\tilde{I} - A_{22})^{-1}[B_2U + A_{21}X + z(0)] - LX \ . \end{aligned} \tag{2.4.12}$$

Take Laplace transform on our manifold expression (2.4.3) and use (2.4.12)

$$\begin{aligned} Z &= LX + P \\ &= LX + (s\tilde{I} - A_{22})^{-1}[B_2U + A_{21}X + z(0)] - LX \\ &= (s\tilde{I} - A_{22})^{-1}[B_2U + A_{21}X + z(0)] \end{aligned}$$

which is the same expression as (2.4.2).

QED

The above theorem justifies the use of the manifold approach in linear time-invariant systems. Note that we can design our controller from the point of view of the integral manifold for those nonlinear systems which are not Laplace transformable and thus renders the frequency domain analysis impossible. The integral manifold design is especially powerful when we are dealing with singularly perturbed systems. A controller in the form of an asymptotic series of ϵ (the perturbation parameter) can be designed based on this methodology. This will be illustrated in the subsequent Sections.

2.5. Eigenvalue Placement Problem

We consider the eigenvalue assignment problem for linear time-invariant systems that contain slow and fast mode eigenvalues. Here slow mode eigenvalues mean eigenvalues with

smaller magnitude as compared relatively with the fast ones or the larger ones of the same system matrix. As will be shown later, this can be done in two stages. First of all, slow mode eigenvalues are brought to the desired ones through feedback of the slow variables. The fast mode assignment is accomplished by applying a fast feedback so that the resultant fast subsystem possesses the desired fast mode eigenvalues. The fast variable is the deviation from the slow manifold which is characterized by the slow eigenspace of the desired slow mode eigenvalues.

Recall from Lemma 2.2.2 that if the linear system

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = A \begin{bmatrix} x \\ z \end{bmatrix} \tag{2.5.1}$$

has a linear integral manifold $z = Lx$, then L is given by

$$L = \begin{bmatrix} v_2 \; v_1^{-1} \end{bmatrix} \tag{2.5.2}$$

where $v_s = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is the slow eigenspace of (2.5.1).

When we want our system (2.5.1) to possess the desired slow mode eigenvalues, i. e., $\{ \lambda_{sd} \}$, through feedback control, the resultant slow eigenspace becomes

$$v_{sd} = \begin{bmatrix} v_{1d} \\ v_{2d} \end{bmatrix} \tag{2.5.3}$$

Apparently the resultant system with feedback has a new shifted manifold given by

$$L_d = (L + M) = \begin{bmatrix} v_{2d} \; v_{1d}^{-1} \end{bmatrix} \tag{2.5.4}$$

where M satisfies (2.3.1) in Lemma 2.3.1. Therefore, assigning slow mode eigenvalues for the linear time-invariant system is equivalent to requiring our closed-loop system to possess the desired slow manifold $z = L_d x$. With Assumptions 2.2.1 and 2.2.2 we shall propose a methodology for our eigenvalue assignment problem.

When our linear system is transformed to an equivalent system using x and η (the deviation from the slow manifold $z = Lx$) as the state variables, the system matrix will be in

upper block triangular form. It is proved in Theorem 2.3.3 that slow manifolds are invariant towards fast feedbacks. In other words, the fast eigenspace (fast manifold) η is orthogonal to the slow eigenspace (slow manifold characterized by $z = Lx$). Hence, when we adopt the slow eigenspace and the fast eigenspace as our coordinates for the linear system we should have a block diagonal matrix as our system matrix. We will name the new coordinates as ρ and η , respectively. It is due to the block diagonal system matrix that we can have the two-stage eigenvalue assignment design.

Recall

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = A \begin{bmatrix} x \\ z \end{bmatrix} + Bu \tag{2.5.5}$$

is equivalent to

$$\begin{bmatrix} \dot{x} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A_s & A_{12} \\ 0 & A_f \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} + \begin{bmatrix} B_1 \\ \tilde{B} \end{bmatrix} u \tag{2.5.6}$$

where

$$A_s = A_{11} + A_{12}L \; , A_f = A_{22} - LA_{12}$$
$$\tilde{B} = B_2 - LB_1 \; , \eta = z - Lx \; ;$$

here, L satisfies (2.2.3). To achieve block diagonalization we have to use ρ , the invariant subspace characterized by $Z = LX$, as our new coordinate . The state variable ρ is obtained by removing the fast eigenspace component in x,

$$\rho = x - H\eta \, , \tag{2.5.7}$$

where H satisfies

$$(A_{11} + A_{12}L)H + A_{12} = H(A_{22} - LA_{12}) \, . \tag{2.5.8}$$

Then (2.5.6) becomes

$$\begin{bmatrix} \dot{\rho} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A_s & 0 \\ 0 & A_f \end{bmatrix} \begin{bmatrix} \rho \\ \eta \end{bmatrix} + \begin{bmatrix} B_1 - H\tilde{B} \\ \tilde{B} \end{bmatrix} u \, . \tag{2.5.9}$$

Equation (2.5.9) without input is in block-diagonal form. When the system starts with its initial condition on the eigenspace spanned by ρ , its motion will be governed by

$$\dot{\rho} = A_s \, \rho$$

and will remain in it as time goes on. This is exactly the description of motion on the slow manifold of the system. Hence, the slow manifold is the invariant subspace spanned by the slow eigenspace of the relevant system matrix . The same argument applies to η , the fast eigenspace, and we shall name it as the fast manifold for the sake of completeness. By assuming the complete controllability of (2.5.5), we also have the complete controllability of the slow and the fast subsystems in (2.5.9). The complete controllability of the slow subsystem in turn ensures that we can choose K so that when

$$u = K\rho + u_2$$

is applied to the linear system, we have

$$\begin{bmatrix} \dot{\rho} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A_s + (B_1 - H\tilde{B})K & 0 \\ \tilde{B}K & A_f \end{bmatrix} \begin{bmatrix} \rho \\ \eta \end{bmatrix} + \begin{bmatrix} B_1 - H\tilde{B} \\ \tilde{B} \end{bmatrix} u_2 \tag{2.5.10}$$

where

$$\lambda_i \left[A_s + (B_1 - H\tilde{B})K \right] = \lambda_{si} \quad , \quad 1 \leqslant i \leqslant n \ ; \tag{2.5.11}$$

$\{ \lambda_{si} \}$ are the desired slow mode eigenvalues.

Since our slow eigenspace has been changed as a result of change of slow eigenvalues, we have a new shifted slow manifold or slow eigenspace characterized by the submatrix

$$A_s + (B_1 - H\tilde{B})K$$

in (2.5.10).

Since we have introduced a slow feedback into the fast subsystem, the fast eigenspace is also changed correspondingly. We shall name the new coordinate spanned by the new fast eigenspace as σ , where

$$\sigma = \eta - N\rho \, , \tag{2.5.12}$$

and N satisfies

$$\tilde{B}K + A_f N = N[A_s + (B_1 - H\tilde{B})K] \, . \tag{2.5.13}$$

Note $\eta = N\rho$ is a slow manifold within the system (2.5.10) without input u_2 .

With (2.5.12)-(2.5.13), we have from (2.5.10)

$$\begin{bmatrix} \dot{\rho} \\ \dot{\sigma} \end{bmatrix} = \begin{bmatrix} A_s + (B_1 - H\tilde{B})K & 0 \\ 0 & A_f \end{bmatrix} \begin{bmatrix} \rho \\ \sigma \end{bmatrix} + \begin{bmatrix} B_1 - H\tilde{B} \\ \tilde{B} - M(B_1 - H\tilde{B}) \end{bmatrix} u_2 . \tag{2.5.14}$$

We still have the complete controllability of the slow and fast subsystem pairs in (2.5.14) since controllability is invariant to state feedback. To achieve the fast eigenvalue placement objective, we pick G so that

$$\lambda_j \{ A_f + G[\tilde{B} - M(B_1 - H\tilde{B})] \} = \lambda_{fj} \quad , \quad 1 \leqslant j \leqslant m \; . \tag{2.5.15}$$

where $\{ \lambda_{fj} \}$ are the desired fast mode eigenvalues. Overall we have achieved the eigenvalue assignment in two stages by applying a composite control that consists of slow and fast feedbacks.

$$u = K\rho + G\sigma \tag{2.5.16}$$

The input can also be expressed in terms of the original state variables x and z.

$$\begin{aligned} u &= K\rho + G\sigma = K\rho + G(\eta - N\rho) \\ &= (K - GN)\rho + G\eta = (K - GN)(x - H\eta) + G\eta \\ &= (K - GN)x + [G - (K - GN)H]\eta \\ &= (K - GN)x + [G - (K - GN)H](z - Lx) \\ &= (K - GN) - [G - (K - GN)H]Lx + [G - (K - GN)H]z \end{aligned} \tag{2.5.17}$$

We now investigate how the slow manifold of the resultant system with input (2.5.16) is characterized in $(x , z)^T$ state space. From (2.5.12) when the system is on the slow manifold

$$\eta = N\rho \, , \tag{2.5.18}$$

eliminating ρ between (2.5.18) and (2.5.7) we have

$$\eta = (I + NH)^{-1}Nx \; . \tag{2.5.19}$$

Substituting this into $z = Lx + \eta$ we have

$$z = Lx + (I + NH)^{-1}Nx = [L + (I + NH)^{-1}N]x \; . \tag{2.5.20}$$

Note that N depends on K but not on G. This again indicates that the shifted linear integral

manifold is invariant towards the fast feedback. Comparing (2.5.20) with (2.3.1) in Lemma 2.3.1 where the shifted manifold is described by Lemma 2.3.1 as

$$z = (L + M)x \ .$$

we have

$$M = (I + NH)^{-1} N \ .$$

From the above discussion we have proved the following theorem.

Theorem 2.5.1

By assuming the complete controllability of the (A , B) pair in

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = A \begin{bmatrix} x \\ z \end{bmatrix} + Bu$$

A composite control of the form

$$u = \alpha x + \beta z$$

will achieve the eigenvalue placement objective, where

$$\beta = G - (K - GN)H$$

$$\alpha = K - GN - \beta L \ .$$

The constants K and G are chosen as in (2.5.11) and (2.5.15). The variables L, H and N satisfy (2.2.3), (2.5.8) and (2.5.13), respectively.

For singularly perturbed systems, we have similar expressions. Furthermore, L, H and N can be approximated by some explicit expressions as follows.

Theorem 2.5.2

To assign { λ_s , λ_f } as the desired eigenvalues to the linear system

$$\begin{bmatrix} \dot{x} \\ \epsilon \dot{z} \end{bmatrix} = A \begin{bmatrix} x \\ z \end{bmatrix} + Bu \tag{2.5.21}$$

where { λ_s } = { λ_{s1} ,...., λ_{sn} } are the desired *distinct* slow eigenvalues, and { λ_f } = { λ_{f1} .... λ_{fm} } are the desired *distinct* fast eigenvalues. Assume the full controllability of the fast and slow subsystem pairs, (A_0 , B_0) and (A_{22} , B_2) , where

$$A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21} \text{ and } B_0 = B_1 - A_{22}^{-1}A_{12}B_1$$

a feedback control of the form

$$u = Kx + G(z - L_d x) = (K - GL_d)x + Gz \tag{2.5.22}$$

with K firstly chosen so that

$$\lambda_i(A_0 + B_0K) = \lambda_{si} \ , \quad 1 \leqslant i \leqslant \imath \ ; \tag{2.5.23}$$

and then G picked so that

$$\lambda_j(A_{22} + B_2G) = \epsilon\lambda_{fj} \ , \quad 1 \leqslant j \leqslant m \ , \tag{2.5.24}$$

will result in

$$\lambda_i^c = \lambda_i(A_0 + B_0K) + O(\epsilon) = \lambda_{si} + O(\epsilon) \ , \quad 1 \leqslant i \leqslant n \tag{2.5.25}$$

$$\lambda_i^c = \lambda_j(A_{22} + B_2G + O(\epsilon))/\epsilon \ , \quad i = n + j \ , \quad 1 \leqslant j \leqslant m \tag{2.5.26}$$

where $\{ \lambda_i^c \}$ are the eigenvalues of the resultant closed-loop system. Also, the L_d in (2.5.22) satisfies

$$A_{21} + B_2K + A_{22}L_d = \epsilon L_d(A_{11} + B_1K + A_{12}L_d) . \tag{2.5.27}$$

Proof:

Apply a composite control of the form

$$u = u_s + u_f = Kx + u_f \tag{2.5.28}$$

to (2.5.21). (2.5.21) then becomes

$$\begin{bmatrix} \dot{x} \\ \epsilon\dot{z} \end{bmatrix} = A \begin{bmatrix} x \\ z \end{bmatrix} + B(u_s + u_f) , \tag{2.5.29}$$

or

$$\begin{bmatrix} \dot{x} \\ \epsilon\dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} + B_1K & A_{12} \\ A_{21} + B_2K & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + Bu_f . \tag{2.5.30}$$

With the assumption that A_{22} is nonsingular, (2.5.30) has a manifold

$$z = L_d x$$

where L_d satisfies (2.5.27) by Lemma 2.2.1 and

$$L_d = L_d^0 + O(\epsilon) = A_{22}^{-1}(A_{21} + B_2K) . \tag{2.5.31}$$

Take η as the deviation of z from the slow manifold $z = L_d x$, i. e.,

$$\eta = z - L_d x$$

We have from (2.5.30),

$$\begin{bmatrix} \dot{x} \\ \epsilon\dot{\eta} \end{bmatrix} = \begin{bmatrix} A_s^K & A_{12} \\ 0 & \tilde{A}^K \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} + \begin{bmatrix} B_1 \\ \tilde{B} \end{bmatrix} u_f \tag{2.5.32}$$

where

$$A_s^K = A_{11} + B_1 K + A_{12} L_d \ , \tilde{A}^K = A_{22} - \epsilon L_d A_{12} \ \text{ and } \tilde{B} = B_2 - \epsilon L_d B_1 .$$

When a fast feedback

$$u_f = G\eta$$

is applied to (2.5.32), we have

$$\begin{bmatrix} \dot{x} \\ \epsilon\dot{\eta} \end{bmatrix} = \begin{bmatrix} A_s^K & A_{12} + B_1 G \\ 0 & \tilde{A}^K + \tilde{B}G \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} .$$

The above system matrix differs by $O(\epsilon)$ from its approximate version

$$\begin{bmatrix} A_0 + B_0 K & A_{12} + B_1 G \\ 0 & A_{22} + B_2 G \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} . \tag{2.5.33}$$

i. e.,

$$A_s^K = A_0 + B_0 K + O(\epsilon) \ \text{ and}$$

$$\tilde{A}^K + \tilde{B}G = A_{22} + B_2 G + O(\epsilon) .$$

By a standard theorem in singular perturbation, the corresponding eigenvalues also differ by $O(\epsilon)$ from its approximate ones [13], i. e.,

$$\lambda_i(A_s^K) = \lambda_i(A_0 + B_0 K) + O(\epsilon) , \quad 1 \leqslant i \leqslant n$$

$$\lambda_j(\tilde{A}^K + \tilde{B}G) = \lambda_j(A_{22} + B_2 G) + O(\epsilon) , \quad 1 \leqslant j \leqslant m .$$

Since (2.5.33) is in block-triangular form and with the controllability assumption we can choose G and K so that (2.5.23) and (2.5.24) are accomplished.

QED

The closed-loop system has a shifted slow manifold

$$z = L_d x = (L + M)x \tag{2.5.34}$$

where L is governed by an equation related to the A_{ij} entries of the open-loop systems. In

other words, L is inherently related to the eigenspace or the eigenvalues of the original system. Altering the eigenvalues would subsequently result in a new eigenspace, which in turn shifts our system to a new manifold (2.5.34). The amount of work required to achieve this through slow feedback is directly related to M, the amount of manifold shifted from the original one. As we can see from (2.3.1), M is a function of both L, an inherent property of the open-loop system, and K, the amount of feedback applied. We now study a *singularly perturbed* system and give an implementation for the eigenvalue sensitivity problem.

Lemma 2.5.1

The singularly perturbed system

$$\begin{bmatrix} \dot{x} \\ \epsilon\dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{2.5.35}$$

with input $u = Kx$ has a linear integral manifold

$$z = \bar{L}x \tag{2.5.36}$$

where $\bar{L}$ satisfies

$$A_{21} + B_2K + A_{22}\bar{L} = \epsilon\bar{L}(A_{11} + B_1K + A_{12}\bar{L})$$

$$\bar{L} = \bar{L}_0 + \epsilon\bar{L}_1 + O(\epsilon) \ ;$$

$$\bar{L}_0 = -A_{22}^{-1}(A_{21} + B_2K_0) \ ; \ \bar{L}_1 = A_{22}^{-1}[\bar{L}_0(A_{11} + B_1K_0 + A_{12}\bar{L}_0) - B_2K_1] \ .$$

With η as the deviation from the slow manifold (2.5.36), (2.5.35) is equivalent to

$$\begin{bmatrix} \dot{x} \\ \epsilon\dot{\eta} \end{bmatrix} = \begin{bmatrix} A_s^k & A_{12} \\ 0 & \tilde{A}^K \end{bmatrix} \begin{bmatrix} x \\ \eta \end{bmatrix} . \tag{2.5.37}$$

$$A_s^K = A_{11} + B_1K + A_{12}\bar{L} \ , \ \tilde{A}_K = A_{22} - \epsilon\bar{L}A_{12}$$

Furthermore, if $\tilde{A}_K$ is Hurwitz for $\epsilon \in (0, \epsilon^*]$ *where* $0 < \epsilon^* << 1$, (2.5.37) will be on the manifold and is equivalent to

$$\dot{x} = A_s^K x \ . \tag{2.5.38}$$

Proof:

Similar to the proof on Theorem 2.5.2.

Note in Lemma 2.5.1. if the fast subsystem on ρ is not stable , i. e., the manifold is repulsive, we can stabilize it by applying an appropriate fast feedback and thus rendering the slow manifold attractive. We are then justified to use (2.5.38). To investigate the eigenvalue sensitivity problem, we need the following Lemma [14].

Lemma 2.5.2

For the linear system

$$\dot{x} = A(\epsilon)x$$

we have the following formula regarding the eigenvalue sensitivity with respect to the parameter ϵ .

$$\frac{\partial \lambda_i}{\partial \epsilon} = \frac{<(\frac{\partial A}{\partial \epsilon})v^i \, , w^i>}{<w^i \, , v^i>} \tag{2.5.39}$$

where $v^i(\epsilon)$ and $w^i(\epsilon)$ are the respective eigenvectors of $A(\epsilon)$, $A^T(\epsilon)$ associated with the eigenvalue λ_i .

Proof:

Take $v^i(\epsilon)$ as the eigenvector of $A(\epsilon)$ associated with the eigenvalue $\lambda_i(\epsilon)$, we have

$$A(\epsilon)v^i(\epsilon) = \lambda_i(\epsilon)v^i(\epsilon) \, .$$

Taking partial derivatives with respect to the parameter ϵ on both sides of the above equation,

$$\frac{\partial A}{\partial \epsilon}v^i + A\frac{\partial v^i}{\partial \epsilon} = \frac{\partial \lambda_i}{\partial \epsilon}v^i + \lambda_i \frac{\partial v^i}{\partial \epsilon} \, . \tag{2.5.40}$$

Left multiply with w_i^T on each terms of (2.5.40),

$$(w^i)^T \frac{\partial A}{\partial \epsilon}v^i + (w^i)^T A\frac{\partial v^i}{\partial \epsilon} = (w^i)^T \frac{\partial \lambda_i}{\partial \epsilon}v^i + (w^i)^T \lambda_i \frac{\partial v^i}{\partial \epsilon} \, . \tag{2.5.41}$$

Observe that since

$$A^T w_i = \lambda_i w_i$$

so we have

$$w_i A = \lambda_i w_i .$$

and also due to the fact that λ is a scalar, (2.5.41) reduces to

$$(w^i)^T \frac{\partial A}{\partial \epsilon} v^i = \frac{\partial \lambda_i}{\partial \epsilon} (w^i)^T v^i .$$

By dividing both sides by $(w^i)^T v^i$, we have (2.5.39).

QED

We are now in the position to propose a design for the eigenvalue placement problem of singularly perturbed systems. The method here is to add a corrective term as part of the feedback to the singularly perturbed system, so that the eigenvalues assigned do not differ more than $O(\epsilon^2)$ from the desired one.

Theorem 2.5.3

For the single input system,

$$\begin{bmatrix} \dot{x} \\ \epsilon \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u$$

if K_0 is the design parameter for the nominal or reduced system,

$$\dot{x} = A_0 x + b_0 u$$

where

$$A_0 = A_{11} - A_{12} A_{22}^{-1} A_{21} ,\ B_0 = B_1 - A_{12} A_{22}^{-1} B_2$$

such that

$$\lambda_i (A_0 + B_0 K_0) = \lambda_{id} \quad , \quad 1 \leq i \leq n$$

and $A_0 + B_0 K_0$ has n linearly independent eigenvectors that span R^n space,

a feedback of the form

$$u = (K_0 + \epsilon K_1) x$$

will place the eigenvalues of the resultant closed-loop system at

$$\lambda_i = \lambda_{id} + O(\epsilon^2) \tag{2.5.42}$$

where $\lambda_{di}, 1 \leq i \leq n$, are the desired eigenvalues.

$$K_1 = PM^{-1} \tag{2.5.43}$$

with

$$M = \left[v_0^1 \,,\, \ldots \,,\, v_0^n \right]$$

where v_0^i are the eigenvectors of $A_0 + B_0K_0$ and

$$P = \left[\frac{<w_0^1 \,.\, A_1 v_0^1>}{<w_0^1 \,.\, B_0>} \,,\, \ldots \,,\, \frac{<w_0^n \,.\, A_1 v_0^n>}{<w_0^n \,.\, B_0>} \right]$$

where <...> stands for dot product. The vectors $w_0^i, 1 \leqslant i \leqslant n$. are the eigenvectors of $(A_0 + B_0K_0)^T$ and

$$A_1 = A_{12}(A_{22}^{-1})^2(A_{21} + B_2K_0)(A_0 + B_0K_0) .$$

We also assume that $< w_0^i \,.\, B_0 > \neq 0$, $1 \leqslant i \leqslant n$ ~.

Proof:

By Lemma 2.5.1 our singularly perturbed system is equivalent to

$$\dot{x} = A_s^K(\epsilon)x$$

if the fast subsystem is stable by itself or through fast feedback somehow.

Note that

$$\begin{aligned} A_s^K(\epsilon) &= A_{11} + B_1K(\epsilon) + A_{12}\bar{L}(\epsilon) \\ &= A_{s0}^K + \epsilon A_{s1}^K + (\epsilon^2) \end{aligned}$$

where

$$\begin{aligned} A_{s0}^K &= A_{11} + B_1K_0 + A_{12}\bar{L}_0 \\ &= A_{11} + B_1K_0 - A_{12}A_{22}^{-1}(A_{21} + B_2K_0) \\ &= (A_{11} - A_{12}A_{22}^{-1}A_{21}) + (B_1 - A_{12}A_{22}^{-1}B_2)K_0 \\ &= A_0 + B_0K_0 \end{aligned}$$

and

$$\begin{aligned} A_{s1}^K &= B_1K_1 + A_{12}\bar{L}_1 \\ &= B_1K_1 + A_{12}A_{22}^{-1}L_0(A_0 + B_0) - A_{12}A_{22}^{-1}B_2K_1 \end{aligned}$$

$$= (B_1 - A_{12}A_{22}^{-1}B_2)K_1 - A_{12}(A_{22}^{-1})^2(A_{21} + B_0K_0)(A_0 + B_0K_0)$$

$$= B_0K_1 - A_{12}(A_{22}^{-1})^2(A_{21} + B_2K_0)(A_0 + B_0K_0)$$

$$= B_0K_1 - A_1 .$$

Applying Lemma 2.5.2,

$$\left(\frac{\partial\lambda_i}{\partial\epsilon}\right)_{\epsilon=0} = \frac{\langle w_0^i , (\frac{\partial A}{\partial\epsilon})_{\epsilon=0} v_0^i \rangle}{\langle w_0^i , v_0^i \rangle} = \frac{\langle w_0^i , A_{s1}^K v_0^i \rangle}{\langle w_0^i , v_0^i \rangle} . \tag{2.5.44}$$

In order that the first term on RHS of (2.5.44) vanishes, we require

$$\langle w_0^i , (B_0K_1 - A_1)v_0^i \rangle = 0 , \quad 1 \leqslant i \leqslant n \tag{2.5.45}$$

or

$$\langle w_0^i , B_0K_1v_0^i \rangle = \langle w_0^i , A_1v_0^i \rangle , \quad 1 \leqslant i \leqslant n$$

or

$$K_1v_0^i = \frac{\langle w_0^i , A_1v_0^i \rangle}{\langle w_0^i , B_0 \rangle} , \quad 1 \leqslant i \leqslant n$$

provided $\langle w_0^i , B_0 \rangle \neq 0$, i. e.,

$$K_1 \left[v_0^1 \; \cdots\cdots \; v_0^n \right] = \left[\frac{\langle w_0^1 , A_1v_0^1 \rangle}{\langle w_0^1 , B_0 \rangle} \; \cdots\cdots \; \frac{\langle w_0^n , A_1v_0^n \rangle}{\langle w_0^n , B_0 \rangle} \right] . \tag{2.5.46}$$

From (2.5.46), we have (2.5.43). By (2.5.44) and (2.5.45) we have (2.5.42).

QED

In the proof of the above theorem, we note that if B_0 is nonsingular, we can take

$$K_1 = B_0^{-1}A_1$$

to achieve the same task. One merit of this kind of corrective measure is that the feedback controller based on the nominal model can still be used. Only an additional term is added to the controller to compensate for the effect of parasitics that is inherent in the singularly perturbed system.

2.6. Application of Slow Manifold to Tracking Problems

We shall investigate the tracking problem of a singularly perturbed system in which the slow part is required to track a prescribed trajectory. When the fast part, i. e., the deviation from the manifold, is asymptotically stable the deviation goes to zero at the rate of O($1/\epsilon$). We can then consider the system as restricted to the slow manifold and thus simplify our design. Tracking of this type arises in many situations. The tracking problem of the flexible link manipulator is just one of such.

By following (2.3.29)-(2.3.31) in Section 2.3,

$$\begin{bmatrix} \dot{x} \\ \epsilon\dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{2.6.1}$$

is equivalent to

$$\begin{bmatrix} \dot{x} \\ \epsilon\dot{p} \\ \epsilon\dot{\eta} \end{bmatrix} = \begin{bmatrix} A_s & A_{12} & A_{12} \\ 0 & \tilde{A} & 0 \\ 0 & 0 & \tilde{A} \end{bmatrix} \begin{bmatrix} x \\ p \\ \eta \end{bmatrix} + \begin{bmatrix} B_1(u + u_f) \\ \tilde{B}u \\ \tilde{B}u_f \end{bmatrix} \tag{2.6.2}$$

where A_s , $\tilde{A}$ and $\tilde{B}$ are as defined, and

$$\begin{aligned} A_s(\epsilon) &= A_{so} + \epsilon A_{s1} + O(\epsilon^2) \\ &= A_{11} + A_{12}L_0 + \epsilon A_{12}L_1 + O(\epsilon^2) \\ &= A_0 + \epsilon A_{12}L_1 + O(\epsilon^2) \end{aligned}$$

and

$$\begin{aligned} \tilde{A} &= \tilde{A}_0 + \epsilon\tilde{A}_1 + O(\epsilon^2) \\ &= A_{22} + \epsilon(-L_0A_{12}) + O(\epsilon^2) \\ &= A_{22} + \epsilon A_{22}^{-1}A_{21}B_1 + O(\epsilon^2) . \end{aligned}$$

It is assumed that the fast variable, η , is available for feedback control. When the fast subsystem of η is stable by itself or somehow through the fast feedback, $u_f = G\eta$, the system will be on the manifold so (2.6.2) is equivalent to

$$\begin{bmatrix} \dot{x} \\ \epsilon \dot{p} \end{bmatrix} = \begin{bmatrix} A_s & A_{12} \\ 0 & \tilde{A} \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} B_1 \\ \tilde{B} \end{bmatrix} u . \tag{2.6.3}$$

Suppose the slow variable is required to track a given trajectory $x_d(t)$, which is assumed to be infinitely differentiable. By setting

$$\dot{x} = v$$

with

$$v = \dot{x}_d - \alpha(x - x_d) \tag{2.6.4}$$

then

$$\dot{e} = -\alpha e ; \tag{2.6.5}$$

where

$$e(t) = x(t) - x_d(t)$$

is the error between the output of the system and that of the desired trajectory at time t. With $\alpha > 0$, we have (2.6.5) as a stable system and x will track $x_d(t)$ as desired.

We can solve p up to any order of ϵ algebraically.

$$\epsilon \dot{p}(\epsilon) = \tilde{A}(\epsilon) p(\epsilon) + \tilde{B}(\epsilon) u(\epsilon)$$

Equating the coefficient of ϵ^0 in the above equation.

$$\epsilon^0 : \ \tilde{A}_0 p_0 + \tilde{B}_0 u_0 = 0 .$$

So

$$p_0 = -\tilde{A}_0^{-1} \tilde{B}_0 u_0 = -A_{22}^{-1} B_2 u_0 .$$

Similarly,

$$\epsilon^1 : \ \dot{p}_0 = \tilde{A}_0 p_1 + \tilde{A}_1 p_0 + \tilde{B}_0 u_1 + \tilde{B} u_0 .$$

So

$$\begin{aligned} p_1 &= A_{22}^{-1} (\dot{p}_0 - \tilde{A}_1 p_0 - \tilde{B}_0 u_1 - \tilde{B}_1 u_0) \\ &= -A_{22}^{-1} (A_{22}^{-1} B_2 \dot{u}_0 + L_0 A_{12} A_{22}^{-1} B_2 u_0 + B_2 u_1 - L_0 B_1 u_0) \\ &= -A_{22}^{-1} (A_{22}^{-1} B_2 \dot{u}_0 - L_0 B_2 u_0 + B_2 u_1) = p_1(u_0, u_1, \dot{u}_0) . \end{aligned}$$

In general,

$$\epsilon^r : \dot{p}_{r-1} = \sum_{i=0}^{r} (\tilde{A}_i p_{r-i} + \tilde{B}_i u_{r-i}) .$$

or

$$p_r = A_{22}^{-1} [\dot{p}_{r-1} - \tilde{B}_0 u_r + \sum_{i=1}^{r} (\tilde{A}_i p_{r-i} + \tilde{B}_i u_{r-i})]$$

$$= A_{22}^{-1} [\dot{p}_{r-1} - B_2 u_r + \sum_{i=1}^{r} L_{i-1} (A_{12} p_{r-i} + B_1 u_{r-i})] ,$$

whence it can be shown that

$$p_r = p_r (u_0 , \ldots , u_r , \dot{u}_0 , \ldots , \dot{u}_{r-1}) . \tag{2.6.6}$$

From (2.6.3), the slow variable is governed by

$$\begin{aligned} \dot{x} &= A_s x + A_{12} p + B_1 u \\ &= (A_{s0} x + A_{12} p_0 + B_1 u_0) + \epsilon (A_{s1} x + A_{12} p_1 + B_1 u_1) + O(\epsilon^2) \\ &= [A_0 x + (B_1 - A_{12} A_{22}^{-1} A_{21}) u_0] + \epsilon (A_{12} L_1 x + A_{12} p_1 + B_1 u_1) \\ &+ \cdots + \epsilon^r (A_{12} L_r x + A_{12} p_r + B_1 u_r) + \cdots \end{aligned} \tag{2.6.7}$$

where

$$A_0 = A_{11} + A_{12} L_0 = A_{11} - A_{12} A_{22}^{-1} A_{21}$$

$$\text{and } B_0 = B_1 - A_{12} A_{22}^{-1} B_2 .$$

We should pick u_0 . so that in the nominal model, i. e., the one obtained by setting the parasitics $\epsilon = 0$.

$$A_0 x + B_0 u_0 = v . \tag{2.6.8}$$

i. e.,

$$u_0 = B_0^{-1} (v - A_0 x) = u_0 (x , x_d , \dot{x}_d) \tag{2.6.9}$$

and u_i so that coefficient of ϵ^r , $1 \leq r < \infty$, in (2.6.7) vanish. Here we assume B_0 to be nonsingular. In this manner the resultant closed-loop system becomes

$$\dot{x} = A_0 x + B_0 u = v . \tag{2.6.10}$$

Or

$$\dot{x} = v .$$

and hence (2.6.5). The tracking objective is thus achieved. We now show how the derivatives

of u_0 are expressed in terms of x, the state of the system, $x_d(t)$ and the higher derivatives of x_d.

From (2.6.8) and (2.6.4)

$$\begin{aligned} u_0 &= B_0^{-1}(v - A_0 x) \\ &= B_0^{-1}(\dot{x}_d - \alpha(x - x_d) - A_0 x) \\ &= B_0^{-1}[\dot{x}_d - \alpha x_d - (A_0 + \alpha I)x]. \end{aligned} \tag{2.6.11}$$

Differentiating both sides of (2.6.11) and use the fact that when an appropriate control of the form

$$u = u_0 + \epsilon u_1 + \cdots$$

is applied to the system on the manifold we have (2.6.10).

$$\begin{aligned} \dot{u}_0 &= B_0^{-1}[\ddot{x}_d - \alpha \dot{x}_d - (A_0 + \alpha I)v] \\ &= B_0^{-1}[\ddot{x}_d - \alpha \dot{x}_d - (A_0 + \alpha I)(\dot{x}_d - \alpha(x - \dot{x}_d))] \\ &= u_0(x, x_d, \dot{x}_d, \ddot{x}_d). \end{aligned}$$

In a similar manner, we have

$$\dot{u}_{r-1} = \dot{u}_{r-1}(x, x_d, \dot{x}_d, \ldots, x_d^{(r+1)}), \quad r \geqslant 1. \tag{2.6.12}$$

Hence, from (2.6.6), (2.6.9) and (2.6.12) we have

$$p_r = p_r(x, x_d, \dot{x}_d, \ldots, x_d^{(r+1)}), \quad r \geqslant 0.$$

Therefore, we should design our control u_r, $1 \leqslant r < \infty$, based on

$$u_r = u_r(x, p_r) = u_r(x, x_d, \dot{x}_d, \ldots, x_d^{(r+1)}).$$

That is to say, to implement the control we need the state of the system, x, the desired trajectory, $x_d(t)$, and its higher derivatives which are assumed to be known a priori. When $\epsilon \to 0$, (2.6.1) reduces to

$$\begin{aligned} z &= -A_{22}^{-1}(A_{21}x + B_2 u) \\ \dot{x} &= A_{11}x + A_{12}z + B_1 u \\ &= (A_{11} - A_{12}A_{22}^{-1}A_{21})x + (B_1 - A_{12}A_{22}^{-1}B_2)u \\ &= A_0 x + B_0 u. \end{aligned} \tag{2.6.13}$$

Thus, for our design

$$u = u_0 + u_c \; .$$

u_0 is the required control for the tracking problem based on the unperturbed or reduced model (2.6.13), and

$$u_c = \epsilon(u_1 + \epsilon u_2 + \cdots)$$

is the corrective control to be added to compensate for the effect of parasitics present in the system. The parasitics in the case of a flexible link manipulator will be the flexibility of the robot arm. Note that by appending a corrective feedback control to the real system the singularly perturbed system will behave, to a naive user, as if it is parasitics free. Figure 2-3 shows the block diagram of the controller which achieves $O(\epsilon^2)$ tracking accuracy. This completes our discussion on the tracking problem using the slow manifold concept.

3. APPLICATION OF INTEGRAL MANIFOLD TO FLEXIBLE LINK MANIPULATORS

3.1. Introduction

There are two main reasons why we want to investigate the control problems of flexible manipulators. First, control algorithms which assume a rigid model for the manipulator are not satisfactory when applied to real robots where perfect rigidity is not a good assumption. Second, most robots are built to be mechanically stiff simply because of the difficulty of controlling the flexible members and not because rigidity is itself inherently attractive. A great deal of research has been devoted to this issue in recent years[6, 15-18].

In this chapter, the flexibility in flexible manipulators is interpreted and shown to be the cause of phase delay in its performance. A phase-lead prefilter is appended to eliminate the error due to flexibility. A time domain analysis using integral manifolds gives an analogous result and provides a simple approximate corrective scheme to the control problems of the flexible link robot.

3.2. Modeling of a Flexible Single Link Manipulator

To demonstrate our principle, we designed a feedback control which, when being applied to the flexible manipulator, results in a performance that is arbitrarily close to the rigid one. In particular, we illustrate our idea by designing a controller that gives the flexible manipulator a performance $O(\epsilon^2)$ close to the rigid one. The small constant $\epsilon = O(\frac{1}{k})$, where k is the flexibility constant. First, derive an approximate model of a single flexible beam as a linearized singularly perturbed system. For convenience we restrict our discussion to a single planar flexible link as shown in Figure 3-1. We assume the mass of the link is uniformly distributed and that gravity acts normal to the plane of motion and thus can be ignored subsequently. We model the flexible planar link as an interconnection of n rigid links, each with length l_i and mass m_i, as shown in Figure 3-2. It is assumed that the links are connected by linear torsional springs, each with stiffness k, and we assume that k is large. The flexible manipulator is thus

represented as a planar n link mechanism. As a consequence, we establish a coordinate system at the base and at each link as shown in Figure 3-2. Let $q_1 q_2, \ldots, q_n$ be the corresponding joint angles measured with respect to these coordinate frames, and let I_i be the moment of inertia of the i th link about the z_i axis which is normal to the plane of motion of the manipulator. For simplicity, we take $I_i = I$, for all $i = 1, \ldots, n$. Then the kinetic energy of the j th link is given by[19]

$$K_j = \frac{1}{2} m_j V_{cj}^T V_{cj} + \frac{1}{2} \omega_j^T I \omega_j \qquad (3.2.1)$$

where ω_j ,the angular velocity of the j th joint, is given by

$$\omega_j = (\dot{q}_1 + \dot{q}_2 + , \ldots, + \dot{q}_j)\hat{z} \qquad (3.2.2)$$

and $\dot{q}_1$ is understood to be the angular velocity of the first joint, etc.

By following the standard derivation of the kinematic motion of any point on the manipulator with rotational joints[19], the velocity of the center of mass of the j th link, V_{cj} , is given by

$$V_{cj} = J_j \dot{q}$$

where

$$\dot{q} = \left[\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n \right]^T \quad \text{and} \quad J_j = \left[J_{j1}, \ldots, J_{jn} \right]$$

J_{ji}, the Jacobian of the center of mass of the j th link with respect to the (i-1) th joint is given by

$$J_{ji} = |\hat{z} \times (\bar{O}_{cj} - \bar{O}_{i-1})| \quad , 1 \leqslant i \leqslant j$$
$$= 0 \quad , j < i \leqslant n .$$

The position of the i th joint, $1 \leqslant i \leqslant n-1$, is

$$\bar{O}_i = \hat{x} \left[l_1 \cos q_1 + l_2 \cos(q_1 + q_2) + , \ldots, + l_i \cos(q_1 + q_2 + , \ldots, + q_i) \right]$$
$$+ \hat{y} \left[l_1 \sin q_1 + l_2 \sin(q_1 + q_2) + , \ldots, + l_i \sin(q_1 + q_2 + , \ldots, + q_i) \right].$$

The position of the center of the mass of the j th link is

$$\bar{O}_{cj} = \bar{r}_{cj} = \hat{x} \left[l_1 \cos q_1 + , \ldots, + l_{j-1} \cos(q_1 + q_2 + , \ldots, + q_{j-1}) + l_{cj} \cos(q_1 + q_2 + , \ldots, + q_j) \right]$$

$$+ \hat{y} \left[l_1 \sin q_1 + \ldots\ldots + l_{j-1} \sin (q_1 + q_2 + \ldots\ldots + q_{j-1}) + l_{cj} \sin (q_1 + q_2 + \ldots\ldots + q_j) \right] .$$

From a symmetry consideration, we take

$$l_j = l = L/n \tag{3.2.3.a}$$

$$m_j = m = M/n \tag{3.2.3.b}$$

$$l_{cj} = l/2 \tag{3.2.3.c}$$

$$I = I_{zz} = ml^2/12 . \tag{3.2.3.d}$$

With L being the length of the undeflected beam, M being the mass of the whole beam, and l_{cj} being the distance of the center of mass of the j *th* link from the j *th* joint. By (3.2.1)-(3.2.3) the total kinetic energy K of the manipulator is then the sum of the individual kinetic energies

$$K = \frac{m}{2} \sum_{j=1}^{n} V_{cj}^T V_{cj} + \frac{I}{2} \sum_{j=1}^{n} \omega_j^T \omega_j$$

$$= \frac{m}{2} \dot{q}^T (\sum_{j=1}^{n} J_j^T J_j) \dot{q} + \frac{I}{2} \dot{q}^T (\sum_{j=1}^{n} E_j^T E_j \frac{1}{j}) \dot{q}$$

$$= \frac{1}{2} \dot{q}^T \left[\sum_{j=1}^{n} (m\, J_j^T J_j + I\, E_j^T E_j \frac{1}{j}) \right] \dot{q}$$

$$= \frac{1}{2} \dot{q}^T M(q) \dot{q} .$$

where

$$E_j = \begin{bmatrix} 1_{(j \times j)} & 0 \\ 0 & 0_{(n-j \times n-j)} \end{bmatrix}$$

$1_{j \times j}$ is a square j × j matrix with all entries being unity.

By restricting our study to slightly flexible manipulators, we have a small deflection along the link and this indicates that

$$(q_2, q_3, \ldots, q_n) = O(\epsilon)$$

where ϵ is a small positive number.

As a matter of fact, with some trigonometric and algebraic simplification,

$$M(q_1, q_2, \ldots, q_n) = M(q_2, q_3, \ldots, q_n) = M_c + O(\epsilon^2) \tag{3.2.4}$$

when we expand it. This means that M(q) is only a function of fast variables, $(q_2, \ldots, q_n)$, and can further be approximated by a constant matrix M_c which is positive definite and symmetric. Since we are mainly interested in designing corrective feedback control up to $O(\epsilon^2)$ accuracy, it is thus acceptable to take M_c instead of M(q) in our derivation that follows. The potential energy P of the manipulator in this case is the sum of the elastic spring potentials

$$P = \frac{k}{2}(q_2^2 + \ldots, + q_n^2) = P(q)$$

where k is the torsional spring constant and q_i are the relevant angular displacements at each fictitious joints. Euler-Lagrange equations are then of the form

$$L = K - P = K(\dot{q}) - P(q)$$

where $K(\dot{q})$ is a valid approximation of the total kinetic energy to $O(\epsilon^2)$ accuracy.

$$\frac{\partial L}{\partial \dot{q}} = M\dot{q} \quad , \quad \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] = M\ddot{q}$$

$$\frac{\partial L}{\partial q_j} = -\frac{\partial P}{\partial q_j} = -kq_j \quad , \; 2 \leqslant j \leqslant n$$

$$= 0 \qquad , \; j = 1 .$$

Assuming that there is a viscous damping term $- d_i \dot{q}_i$ at the joints, the system equation that describes the flexible beam can be written as

$$M(q)\ddot{q} + \begin{bmatrix} 0 & 0 \\ 0 & I_{n-1} \end{bmatrix} k\, q + d\dot{q} = \begin{bmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} u . \tag{3.2.5}$$

By the fact that M is nonsingular and can be approximated by a constant matrix as in (3.2.4), (3.2.5) is equivalent to the following with $O(\epsilon^2)$ approximation.

$$\ddot{q} = D\dot{q} + A\; kq + Bu \tag{3.2.6}$$

where D, A, B are all constant matrices

By scaling $(q_2, \ldots, q_n) = (\hat{\epsilon} z_2, \ldots, \hat{\epsilon} z_n) = \hat{\epsilon} z$ with $\hat{\epsilon} = 1/k$, we have $q = (q_1, \ldots, q_n)^T = (x, \hat{\epsilon} z)^T$, where $x = q_1$. Without loss of generality, we take $\hat{\epsilon} = \epsilon$. Equation (3.2.6) becomes

$$\begin{bmatrix} \ddot{x} \\ \epsilon \ddot{z} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} \dot{x} \\ \epsilon \dot{z} \end{bmatrix} + \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} z + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \,. \tag{3.2.7}$$

The rigid or reduced model can be deduced from (3.2.7) by setting $\epsilon = 0$. This is equivalent to having an infinitely stiff beam which is also the undeflected rigid beam. Setting $\epsilon = 0$ gives

$$\hat{z} = -A_2^{-1} (D_{21}\dot{x} + B_2 u) \,.$$

Substitute this into (3.2.5) and simplify

$$\ddot{x} = (D_{11} - A_1 A_2^{-1} D_{21})\dot{x} + (B_1 - A_1 A_2^{-1} B_2) u$$

Or

$$\ddot{x} = A_r \dot{x} + B_r u \,. \tag{3.2.8}$$

(3.2.8) can be compared with the system equation from the rigid beam derivation. They are found to be identical.

A more realistic modeling of flexible link manipulators can be obtained by using the modal analysis. A flexible link manipulator with a concentrated mass is shown in Figure 3-3. The deflection along the flexible link, $\delta(x, t)$, is a function of both time and position along the link. $0 \leqslant x \leqslant l$, where l is the length of the undeflected link. The deflection $\delta(x, t)$ is governed by the beam equation with the boundary conditions where the beam is clamped at one end and free at the other end.

$$EI \frac{\partial^4 \delta}{\partial x^4} = -\rho \frac{\partial^2 \delta}{\partial t^2} \tag{3.2.9}$$

where the constants are, respectively,

E: Young's Modulus,

I: beam area inertia,

ρ: density.

With the assumed mode method, $\delta(x, t)$ can be represented as an infinite series of separable modes

$$\delta(x, t) = \sum_{i=1}^{\infty} \pi_i(x)\phi_i(x) \qquad (3.2.10)$$

where $\phi_i(t) \to 0$ as $i \to \infty$ [20]. For a realistic representation of a slightly flexible beam, a good approximation can be obtained by truncating (3.2.10) after the first few terms. The functions $\pi_i(x)$ are the eigenfunctions of the PDE (3.2.9) and satisfy

$$EI \frac{\partial^4 \pi_i}{\partial x^4} = \omega_i^2 \rho \pi_i(x)$$

We treat ϕ_i as part of the generalized coordinate $q = (\theta, \phi)^T$, where θ is the joint angle of the relevant link. Through Lagragian formulation the flexible link manipulator can be modeled by the following state equations[21]:

$$M(q)\ddot{q} + \begin{bmatrix} 0 & 0 \\ 0 & I_{n-1} \end{bmatrix} k\, q + d\dot{q} = Q\, u \qquad (3.2.11)$$

where $q = (\theta, \phi)$ and Q is a constant vector. The $d\dot{q}$ term accounts for damping. Constant k is a normalized stiffness constant that arises as a result of the presence of link flexibility and is related to the payload mass, length, cross-section area, cross-area inertia, density, and the Young's modulus of the beam. A quick comparison between (3.2.5) and (3.2.11) reveals the fact that both ways of modeling flexible link manipulators lead to two equivalent system representations, though there is no one-one correspondence of the state variables between them. For a single flexible beam with no payload that has its motion restricted to a horizontal plane [18], it can be verified that M(q) is a function of fast variables (deflection variables) ϕ_i alone and can further be approximated by a constant matrix as in (3.2.4). This again justifies our way of modeling the flexible beam as n sublinks each connected to the other through a stiff spring except at the base where it is hinged upon a rigid joint. Neglecting the damping effect, Judd and Falkenburg used the Denavit-Hartenberg 4-parameter representation for modeling the flexible beam and came up with a set of system equations identical to (3.2.11) with the

damping term omitted [22].

3.3. Existence of Integral Manifold in Flexible Link Robot System

The flexible manipulator system (3.2.7) can be rewritten as the following singularly perturbed linear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \mu\dot{z}_1 \\ \mu\dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & D_{11} & A_1 & \mu D_{12} \\ 0 & 0 & 0 & I \\ 0 & D_{21} & A_2 & \mu D_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ b_1 \\ 0 \\ b_2 \end{bmatrix} u \tag{3.3.1}$$

where $\mu = \epsilon^{1/2}$, $x_1 = x$, $x_2 = \dot{x}$, $z_1 = z$, and $z_2 = \mu\dot{z}$

This can be rewritten in a more compact form as

$$\begin{bmatrix} \dot{X} \\ \mu\dot{Z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{3.3.2}$$

where $X = (x_1 , x_2)$, $Z = (z_1 , z_2)$ and A_{ij} and B_i correspond to appropriate entries in (3.3.1).

Due to the nonsingularity of M(q), A_2 is nonsingular and hence

$$A_{22}^{-1} = \begin{bmatrix} 0 & I \\ A_2 & \mu D_{22} \end{bmatrix}^{-1} = \begin{bmatrix} -\mu A_2^{-1} D_{22} & A_2^{-1} \\ I & 0 \end{bmatrix} . \tag{3.3.3}$$

By (3.3.3) the existence of a conditionally attractive manifold for (3.3.2) is assured [11]. From Corollary 2.2.1 of Chapter 2, the integral manifold is of the form

$$Z = LX + P$$

where $X = (x_1 , x_2)^T$, $Z = (z_1 , z_2)^T$ and L is a 2×2 constant matrix.

By Corollary 2.2.1, L satisfies the following equation:

$$\begin{bmatrix} 0 & 0 \\ 0 & D_{21} \end{bmatrix} + \begin{bmatrix} 0 & I \\ A_2 & \mu D_{22} \end{bmatrix} L = \mu L \left\{ \begin{bmatrix} 0 & I \\ 0 & D_{11} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ A_1 & \mu D_{12} \end{bmatrix} L \right\}$$

Solving

$$L = \begin{bmatrix} 0 & -\mu A_2^{-1} D_{21} \\ 0 & -A_2^{-1} D_{21} (D_{11} - A_1 A_2^{-1} D_{21}) \end{bmatrix} + O(\mu^2)$$

and P satisfies

$$\mu \dot{P} = \tilde{A} P + \tilde{B} u$$

with $\tilde{A} = A_{22} - \mu L A_{12}$ and $\tilde{B} = B_2 - \mu L B_1$.

For our system to start on manifold $Z = LX + P$, it is necessary and sufficient that

$$Z(t_0) = LX(t_0) + P(t_0) = LX(t_0) \tag{3.3.4}$$

since $P(t_0) = 0$.

We recall that (X, Z) represents the positions and velocities of the joints and deflections, respectively. One of the initial conditions that would satisfy (3.3.4) is the one where the robot starts from rest with undeflected links in the "zero" position. In this case, we have $(X(t_0), Z(t_0)) = 0$ and (3.3.4) is trivially satisfied.

3.4. Flexibility as a Cause for Phase Delay

It is intuitive that for a robot with perfect rigidity the links and the end-effector will move accordingly when the motor starts running as a result of the applied input torque. However, for a robot with flexibility in either links or joints, the end-effector does not move simultaneously with the motor. Furthermore, the trajectory of the robot arm does not follow the expected one exactly. This phenomenon becomes more noticeable with the increase of input frequency that in turn excites the inherent flexible modes in the robot. We shall interpret this as a phase delay due to the flexibility in the manipulators.

Referring to the flexible beam in Figure 3-3, the position of the mass m at the tip of the beam is described by

$$y = l\theta(t) + \delta(l, t),$$

where $l\theta$ is the arc traversed by the tip of the undeflected beam from the reference frame and $\delta(l, t)$ is the deflection at the tip of the beam.

By (3.2.10), (3.3.1) and the fact that we take z_1 as the scaled version of the deflection vari-

ables ϕ , i. e., $\epsilon z_1 = \phi$, we have

$$y = l\,\theta(t) + \sum_{i=2}^{n} \pi_i(l)\phi_i(t)$$

$$= l x_1 + \epsilon(\pi_2(l), \pi_3(l), \cdots, \pi_n(l))\, z_1$$

Since $X = (x_1, x_2)^T$, $Z = (z_1, z_2)^T$, we have

$$y = C_0 X + \epsilon C_1 Z \tag{3.4.1}$$

$$= C_0 X + \mu^2 C_1 Z$$

where $C_0 = (l\ , 0)$, $C_1 = (\pi_2(l), \cdots, \pi_n(l), | 0, \cdots, 0)$, $\epsilon = \mu^2$

Thus, we have the movement of the mass m at the tip of the flexible beam governed by a linear time-invariant system (3.3.2) with linear output (3.4.1).

The movement of the mass m on a rigid beam is described by the following system equations:

$$\dot{X} = A_0 X + B_0 u \tag{3.4.2}$$

$$y = C_0 X \tag{3.4.3}$$

$$A_0 = A_{11} - A_{12} A_{22}^{-1} A_{21}\ , B_0 = B_1 - A_{12} A_{22}^{-1} B_2$$

where (3.4.2) and (3.4.3) are obtained by setting $\mu = 0$ in (3.3.2) and (3.4.1) respectively.

The frequency domain representation of (3.4.2)-(3.4.3) is given by

$$Y = C_0(sI - A_0)^{-1} B_0 U = C_0(sI - A_{11} + A_{12} A_{22}^{-1} A_{21})^{-1} B_0 U\ . \tag{3.4.4}$$

The flexible beam system (3.3.2) as restricted to the manifold $Z = LX + P$ is

$$\dot{X} = (A_{11} + A_{12} L) X + A_{12} P + B_1 u \tag{3.4.5}$$

$$\mu \dot{P} = \tilde{A} P + \tilde{B} u\ . \tag{3.4.6}$$

The frequency domain representation of the movement of mass m on the flexible beam described by (3.4.5)-(3.4.6) is given by

$$Y(s) = C_0 X(s) + \mu^2 C_1 (LX(s) + P(s))$$

$$= C_0 (sI - A_{11} - A_{12} L)^{-1} (B_1 + A_{12}(\mu s I - \tilde{A})^{-1} \tilde{B})\, U$$

$$+\mu^2 C_1(LC_0(sI - A_{11} - A_{12}L)^{-1}(B_1 + A_{12}(\mu sI - \tilde{A})^{-1}\tilde{B}) + (\mu sI - \tilde{A})^{-1}\tilde{B})U$$

Note that

$$(\mu sI - \tilde{A})^{-1} = -(I + \mu s\tilde{A}^{-1})\tilde{A}^{-1} + O(\mu^2)$$

So

$$Y = C_0(sI - A_{11} - A_{12}L)^{-1}(B_1 - A_{12}(I + \mu s\tilde{A}^{-1})\tilde{A}^{-1}\tilde{B})U + O(\mu^2) \qquad (3.4.7)$$

where, as a reminder, $\tilde{A} = A_{22} + O(\mu)$, $\tilde{B} = B_2 + O(\mu)$, $L = -A_{22}^{-1}A_{21} + O(\mu)$

As $\mu \to 0$ the flexible beam reduces to a rigid beam and consequently (3.4.7) becomes that of (3.4.4). One can check this easily by setting $\mu = 0$ in (3.4.7). Comparing (3.4.7) with (3.4.4), we found the perturbation parameter μ, which arises due to the presence of flexibility, induces a phase delay term $\mu s\tilde{A}^{-1}$ in the flexible system output.

To illustrate the principle, we consider a scalar singularly perturbed system similar to (3.3.2) that describes the flexible beam:

$$\dot{x} = ax + bz \qquad (3.4.8.a)$$

$$\mu\dot{z} = -z + u \qquad (3.4.8.b)$$

where $0 < \mu << 1$, and x, z, and u are scalars. Equation (3.4.8) is a simplified version of (3.3.2). We shall name (3.4.8) as the flexible model with μ representing the stiffness constant. The rigid model is obtained when we let $\mu \to 0$. This results in $z = u$ in (3.4.8.b). Substituting this into (3.4.8.a) gives the rigid model equation

$$\dot{x} = ax + bu \ . \qquad (3.4.9)$$

- **Frequency domain analysis**

The Laplace transform of (3.4.8) with zero initial condition is

$$X(s) = \frac{b}{(s-a)(1+\mu s)}U(s) \qquad (3.4.10)$$

where $s = j\omega$. ω represents frequency. Functions X(s) and U(s) are the Laplace transform of x(t) and u(t) respectively.

When $\mu \to 0$, i. e., the spring constant $\to \infty$, (3.4.10) reduces to

$$X(s) = \frac{b}{(s-a)} U(s) \tag{3.4.11}$$

which is precisely the frequency domain representation of the rigid model (3.4.9).

By comparing (3.4.10) with (3.4.11) one would readily see that there is a phase delay of $\tan^{-1}(\mu\omega)$ in the output performance of a flexible robot if a control based on rigid model assumption is applied to it. It is easy to see that phase delay will increase with the increase of input frequency. This explains why in a slightly flexible robot, high-speed performance is usually not satisfactory, though it is fairly acceptable at low-speed maneuvers. The Taylor series expansion of (3.4.10) with respect to μ is

$$\frac{b}{(s-a)(1+\mu s)} = \frac{b\,(1-\mu s)}{(s-a)} + O(\mu^2) \tag{3.4.12}$$

Neglecting $O(\mu^2)$ terms in (3.4.12) and comparing it with the rigid model (3.4.11), we notice that there is an additional unstable zero in the flexible robot system. This is first observed in [23]. Thus we have shown that flexibility not only causes a phase delay but also induces an unstable zero in the system.

In order that the behavior of the flexible model be like a rigid one in the mid-frequency range where $O((\mu s)^2)$ is negligible, we must use a control $\bar{u}$ such that

$$\frac{b}{(s-a)} U(s) = \frac{b\,(1-\mu s)}{(s-a)} \bar{U}(s) .$$

Whence

$$\bar{U}(s) = \frac{U(s)}{(1-\mu s)} . \tag{3.4.13}$$

Note that $\bar{U}(s)$ is obtained by sending the rigid control through a phase-lead compensator.

- **Time domain analysis using integral manifold**

By Corollary 2.3.2, (3.4.8) possesses an attractive input dependent integral manifold of the form $z = p$, where p satisfies

$$\mu \dot{p} = -p + u \ .$$

With

$$x = x_0 + \mu x_1 + \mu^2 x_2 + \dots$$

$$p = p_0 + \mu p_1 + \mu^2 p_2 + \dots$$

$$u = u_0 + \mu u_1 + \mu^2 u_2 + \dots$$

We have

$$p_0 = u \ , \ p_1 = u_1 - \dot{u}_0 \ , \ \cdots \ ,$$

So

$$\dot{x} = ax + bu_0 + \mu b \, (u_1 - \dot{u}_0) + O(\mu^2) \ . \qquad (3.4.14)$$

Note that at $\mu = 0$, (3.4.14) reduces to the rigid model (3.4.9) and hence we are justified that u_0 is the rigid control applied to the ideal rigid model.

By appending a corrective control , $u_1 = \dot{u}_0$, to the nominal rigid controller, the flexible model will behave like the rigid one in the mid-frequency range where $O((\mu s)^2)$ is negligible.

In other words, applying

$$\bar{u} = u_0 + \mu u_1 = u_0 + \mu \dot{u}_0 \qquad (3.4.15)$$

to the realistic flexible system will make the flexible system performance identical to the rigid one in the mid-frequency range.

Note that (3.4.15) in the frequency domain is

$$\bar{U} = (1 + \mu s) U_0$$

which is equivalent to (3.4.13) in the mid-frequency range where $O((\mu s)^2)$ is negligible.

The same argument can be applied to the flexible link system (3.3.2) and similar result can be obtained.

We now recapitulate what we have done. A case study of a flexible beam system reveals that flexibility causes phase delay and thereby deteriorates its expected performance based on a rigid model assumption. The frequency domain analysis comes up with a corrective scheme

which is equivalent to the the time domain approach. where the integral manifold idea is used as a design tool.

3.5. Case Study of a Mechanical System with Flexible Interconnection

In previous sections we investigated the control issue of flexible link robots where perfect rigidity is assumed for the joints. Due to the deformation of gear teeth or bearings within the joints. we also have to face the control problems of manipulators with elastic joints. Spong. Khorasani. and Kokotovic model the joint flexibility in the rigid link manipulator by introducing a fictitious stiff spring within the joint [6]. The motor shaft is interconnected to the relevant link through this spring. which becomes a rigid connection as the spring constant tends to infinity. In this section. we shall study the modeling of a mechanical system of two interconnected masses. which is similar in principle to the elastic joint modeling in [6].

Consider the mechanical system in Figure 3-4 where M is attached to a reference frame through a spring with spring constant k_s and m is driven by an external force f. Masses M and m are interconnected by a stiff spring with spring constant k_f , and x and z are the displacement associated with M and m. respectively. Viscous damping is modeled by the damping constants D and B. respectively.

The equation of motion for this mechanical system can be written as

$$M\ddot{x} = -k_s x - k_f (x - z) - B(\dot{x} - \dot{z}) + Mg - D\dot{x} \tag{3.5.1.a}$$

$$m\ddot{z} = k_f (x - z) + B(\dot{x} - \dot{z}) + mg + f \tag{3.5.1.b}$$

where (.) and (..) represent the first and second derivatives with respect to time. and g is the gravitational constant.

Dividing both sides of (3.5.1) by mass

$$\ddot{x} = -\frac{k_s}{M}x - \frac{k_f}{M}(x - z) - \frac{B}{M}(\dot{x} - \dot{z}) + g - \frac{D}{M}\dot{x} \tag{3.5.2.a}$$

$$\ddot{z} = \frac{k_f}{m}(x - z) + \frac{B}{m}(\dot{x} - \dot{z}) + g + \frac{f}{m} . \tag{3.5.2.b}$$

Introducing a new state variable

$$y = k_f (x - z) \ , \quad \text{and} \quad k_f = \frac{1}{\mu^2} \ .$$

equation (3.5.2.a) becomes

$$\ddot{x} = -\frac{k_s}{M}x - \frac{1}{M}y - \mu^2 \frac{B}{M}\dot{y} + g - \frac{D}{M}\dot{x} \ . \tag{3.5.3.a}$$

By subtracting (3.5.2.b) from (3.5.2.a), we get

$$\mu^2 \ddot{y} = -\frac{k_s}{M}x - (\frac{1}{M} + \frac{1}{m})y - \mu^2 B (\frac{1}{m} + \frac{1}{M})\dot{y} - \frac{D}{M}\dot{x} - \frac{f}{m} \ . \tag{3.5.3.b}$$

To transform our system into a standard singularly perturbed linear system, we use

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \ , \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y \\ \mu \dot{y} \end{bmatrix} .$$

With this, (3.5.3) becomes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \mu \dot{y}_1 \\ \mu \dot{y}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_s}{M} & -\frac{D}{M} & \frac{-1}{M} & -\mu \frac{B}{M} \\ 0 & 0 & 0 & 1 \\ \frac{-k_s}{M} & -\frac{D}{M} & -\left[\frac{m + M}{mM}\right] & -\mu B \left[\frac{m + M}{mM}\right] \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g \\ 0 \\ -\frac{f}{m} \end{bmatrix} \tag{3.5.4}$$

For a stiff spring k_f is a large constant and this implies μ is a small constant, which assures the singularly perturbed form in (3.5.4). As $k_f \to \infty$, *or* $\mu \to 0$, our mechanical system becomes the one with a rigid connection and (3.5.4) reduces to

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k_s}{m + M} & -\frac{D}{m + M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ g + \frac{f}{m + M} \end{bmatrix} \tag{3.5.5}$$

with the use of equalities

$$y_2 = 0 \ , \ y_1 = -\frac{mM}{m + M} (\frac{k_s}{M}x_1 + \frac{D}{M}x_2 + \frac{f}{m})$$

obtained by setting $\mu = 0$ in (3.5.4).

Equation (3.5.5) is the system description for the rigidly connected mechanical system as shown in Figure 3-5, where we have a single object with mass $m + M$ attached to the inertial

frame through a spring with spring constant k_s and damping constant D.

With $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ and $Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$ (3.5.4) can be written as

$$\begin{bmatrix} \dot{X} \\ \mu \dot{Y} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix} \tag{3.5.6}$$

where $u = \begin{bmatrix} 0 \\ g \end{bmatrix}$, $v = \begin{bmatrix} 0 \\ -\frac{f}{m} \end{bmatrix}$ and

$$A_{22} = \begin{bmatrix} 0 & 1 \\ -\frac{m+M}{mM} & -\mu B \left[\frac{m+M}{mM}\right] \end{bmatrix} \quad \textit{is nonsingular} .$$

In fact

$$A_{22}^{-1} = \begin{bmatrix} -\mu B & -\frac{mM}{m+M} \\ 1 & 0 \end{bmatrix} .$$

By Corollary 2.2.2 there exists an integral manifold $Z = LX + P$ for (3.5.6). The constant L and the variable P satisfy, respectively,

$$A_{21} + A_{22}L = \mu L \,(A_{11} + A_{12}L) \tag{3.5.7.a}$$

$$\mu \dot{P} = (A_{22} - \mu L A_{12})P + (v - \mu L u) \tag{3.5.7.b}$$

with

$$L = L_0 + \mu L_1 + \mu^2 L_2 + \cdots$$

$$P = P_0 + \mu P_1 + \mu^2 P_2 + \cdots$$

Equation (3.5.6) as restricted to the manifold becomes

$$\begin{aligned} \dot{X} &= (A_{11} + A_{12}L)X + (u + A_{12}P) \\ &= (A_{11} + A_{12}L_0)X + (u + A_{12}P_0) \\ &\quad + \mu(A_{12}L_1X + A_{12}P_1) + \mu^2(A_{12}L_2X + A_{12}P_2) + \cdots \end{aligned} \tag{3.5.8}$$

Consider the case where f is a constant force, such as a mass of weight f, then P can be solved by equating coefficients of different powers of μ on both sides of (3.5.7)

$$P_0 = -A_{22}^{-1}v$$

$$P_1 = A_{22}^{-1}(\dot{P}_0 + L_0A_{12}P_0 + L_0u)$$

$$= A_{22}^{-1}(-A_{22}^{-1}\dot{v} - L_0A_{12}A_{22}^{-1}v + L_0u)$$

$$P_2 = A_{22}^{-1}(\dot{P}_1 + L_1A_{12}P_0 + L_0A_{12}P_1 + L_1u).$$

With the assumption that f is a constant, u and v will also be constants. So we have

$$P_0 = -A_{22}^{-1}v$$

$$P_1 = A_{22}^{-1}L_0(-A_{12}A_{22}^{-1}v + u)$$

$$P_2 = A_{22}^{-1}(L_1A_{12}P_0 + L_0A_{12}P_1 + L_1u)$$

where

$$L_0 = -A_{22}^{-1}A_{21}$$

$$L_1 = A_{22}^{-1}L_0(A_{11} + A_{12}L_0)$$

$$L_2 = A_{22}^{-1}(L_1(A_{11} + A_{12}L_0) + L_0A_{12}L_1) \quad \cdots$$

as solved from (3.5.7.a).

Substituting A_{ij} by corresponding entries from our mechanical system description (3.5.4), we have the relevant coefficients of μ^0 terms:

$$A_{11} + A_{12}L_0 = \begin{bmatrix} 0 & 1 \\ -\dfrac{k_s}{m+M} & -\dfrac{D}{m+M} \end{bmatrix}$$

$$u + A_{12}P_0 = \begin{bmatrix} 0 \\ g + \dfrac{f}{m+M} \end{bmatrix}$$

$$L_0 = \begin{bmatrix} -\dfrac{mk_s}{m+M} & -\dfrac{mD}{m+M} \\ 0 & 0 \end{bmatrix}.$$

μ^1 terms:

$$A_{12}L_1 = A_{12}A_{22}^{-1}L_0(A_{11} + A_{12}L_0)$$

$$= 0 \cdot (A_{11} + A_{12}L_0) = 0$$

$$A_{12}P_1 = 0.$$

μ^2 terms:

$$A_{12}L_2 = \begin{bmatrix} 0 & 0 \\ \dfrac{m^2 k_s (k_s (m+M) - D^2)}{(m+M)^4} & \dfrac{m^2 D (2k_s (m+M) - D^2)}{(m+M)^4} \end{bmatrix}$$

$$A_{12}P_2 = \begin{bmatrix} 0 & 0 \\ g + \dfrac{f}{m+M} & \dfrac{m^2(k_s (m+M) - D^2)}{(m+M)^3} \end{bmatrix} .$$

Thus, (3.5.8) can be rewritten as

$$\ddot{x} = q_1(\mu)\left[-\frac{k_s}{m+M}x\right] + q_2(\mu)\left[-\frac{D}{m+M}\dot{x}\right] + q_1(\mu)\left[g + \frac{f}{m+M}\right] \tag{3.5.9}$$

with $O(\mu^3)$ terms neglected and

$$q_1(\mu) = 1 + \mu^2 \frac{m^2(k_s(m+M) - D^2)}{(m+M)^3}$$

$$q_2(\mu) = 1 + \mu^2 \frac{m^2 D(2k_s(m+M) - D^2)}{(m+M)^4} .$$

By using a scaled variable $x_\mu = x / q_1(\mu)$ (3.5.9) becomes

$$\ddot{x}_\mu = -\frac{q_1(\mu)k_s}{m+M}x_\mu - \frac{q_2(\mu)D}{m+M}\dot{x}_\mu + \left[g + \frac{f}{m+M}\right] . \tag{3.5.10}$$

Note that with $\mu = 0$ (3.5.10) reduces to the mechanical system with a rigid connection as described by (3.5.5), or

$$\ddot{x} = -\frac{k_s}{m+M}x - \frac{D}{m+M}\dot{x} + \left[g + \frac{f}{m+M}\right]$$

with natural frequency $\omega_0 = \sqrt{\dfrac{k_s}{m+M}}$ and damping ratio $\xi_0 = \dfrac{1}{2}\sqrt{\dfrac{D^2}{k_s m + M}}$.

On the other hand, the perturbed system (3.5.10), i. e., the one with a flexible connection, can be viewed as the one with a displaced *center of mass*, $x_\mu = x / q_1(\mu)$, and

perturbed spring constant: $q_1(\mu)k_s$.

perturbed damping ratio: $q_2(\mu)D$.

natural frequency: $\omega_\mu = \sqrt{\dfrac{k_s q_1(\mu)}{m + M}}$,

damping ratio: $\xi_\mu = \dfrac{1}{2} \sqrt{\dfrac{D^2 q_2^2(\mu)}{k_s q_1(\mu)(m + M)}}$.

The connected system as a whole can be viewed as the one in Figure 3-6 with the above characteristic constants. Suppose we have an underdamped system in the rigidly connected case, i. e.,

$0 < \xi_0 < 1$ or $0 < \dfrac{D^2}{(m + M)k_s} < 4$.

For M and m both large enough as compared with damping constant D, we have $q_1(\mu) > 1$ and $q_2(\mu) > 1$. This implies that the mechanical system as a whole has an increased natural frequency and damping ratio due to the presence of flexibility, i. e., $\mu \neq 0$.

3.6. Conclusion

Additional *fast* states are introduced to take into account the presence of flexibility in the manipulators. The resultant system is a singularly perturbed version of the rigid model equation. Flexible link robots are shown to be in this singularly perturbed form, and the system equations possess an integral manifold. We indicated and proved that the flexibility is a cause of phase delay which induces unsatisfactory performance in the nonrigid robot with a presumed rigid modeling. Frequency domain analysis and time domain analysis using the idea of the integral manifold both come up with the same remedy scheme which demands an additional corrective control be appended to the nominal controller to compensate for the phase delay. Last, we extended our idea to an interconnected mechanical system which contains flexible joint robots as a special case. The system with a nonrigid connection is shown to have a perturbed natural frequency and damping ratio and a displaced *center of mass* from that of the rigidly connected one.

4. TRACKING AND DISTURBANCE REJECTION IN NONLINEAR SYSTEMS BY NONLINEAR INTEGRAL CONTROL

4.1. Introduction

We shall investigate the tracking and disturbance rejection problem of a class of time-invariant nonlinear systems which are *linear equivalent to controllable linear systems*. By using a nonlinear feedback control and a slowly varying integral control the closed-loop system asymptotically tracks a reference input and rejects disturbances which are both *unknown and slowly varying*. The *Integral manifold* concept will be used to design a nonlinear integral controller.

The so-called PI controllers have been used extensively for asymptotic tracking of constant but unknown set-point and rejection of constant disturbances. For *linear time-invariant controllable* systems with nonlinear output, Smith and Davison showed that a full state feedback plus an integral control are needed to achieve the asymptotic tracking and disturbance rejection with the resultant closed-loop system remaining stable [24]. By using a small integral gain Kokotovic pointed out that in a linear time-invariant system the effect of the integral control is to counteract the disturbance terms [13]. For nonlinear systems Desoer and Lin proved that a PI controller can be used to asymptotically track reference inputs and reject disturbances provided that the given system is *exponentially stable* and has a strictly increasing dc steady-state I/O map [25].

In this chapter we study the tracking and disturbance rejection for a class of nonlinear systems which are equivalent to linear controllable systems through a diffeomorphism of change of coordinates and external feedback linearization. Once the given system is transformed to its linear equivalent, it is first of all stabilized by using a full state feedback. A slowly varying integral control is then applied for the purpose of disturbance rejection and asymptotic tracking. The overall system consists of a fast linear subsystem governing the states of the given plant, and a slow nonlinear differential equation governing the variation of the integral control. We will show that there exists an integral manifold for the overall

system. By stabilizing the linear equivalent system the integral manifold for the overall system is rendered attractive. Finally we show that when the system reaches the equilibrium somewhere on the slow manifold, asymptotic tracking of reference input is then accomplished.

We start by reviewing some concepts in differential geometry and external feedback linearizaton, and also by giving a description of our problem. To facilitate our discussion we shall consider the set-point problem alone in Section 4.4. Later, with some additional assumptions, we continue our analysis in Section 4.5 when an unknown but constant disturbance is also present. The result is extended to slowly varying unknown reference input and disturbance in Section 4.6. Finally, in Section 4.7, we illustrate our methodology by a second-order example which is unstable and has a nonzero output at the origin.

4.2. Some Useful Concepts of Differential Geometry

The *Lie bracket* of two C^∞ vector fields on R^n, f and g, is defined by

$$\left[f \ , g \right] \equiv \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

where $\frac{\partial g}{\partial x}$ and $\frac{\partial f}{\partial x}$ are n × n Jacobian matrices and $\left[f \ , g \right]$ is itself a vector field on R^n.

Successive *Lie brackets* are denoted by

$$ad_f^k (g) \equiv \left[f \ , ad_f^{k-1}(g) \right]$$

$$ad_f^0 (g) \equiv g$$

A set of C^∞ vector fields $\{f_1, \ldots, f_m\}$ on R^n is said to be involutive if there exist C^∞ functions $\alpha_{ijk}(x)$ such that

$$\left[f_i \ , f_j \right](x) = \sum_{k=1}^{m} \alpha_{ijk}(x) f_k(x) \qquad 1 \leqslant i \ , j \leqslant m \ .$$

Suppose $f_1, \ldots, f_m$ are linearly independent on R^n. $\{f_1, \ldots, f_m\}$ is said to be completely integrable if there exists an m-dimensional submanifold M in R^n such that at each point of M the tangent space of M is spanned by $\{f_1, \ldots, f_m\}$.

For Ω: $R^n \rightarrow R$, the gradient of Ω is a row vector

$$d\,\Omega = \left(\frac{\partial \Omega}{\partial x_1}, \ldots, \frac{\partial \Omega}{\partial x_n} \right).$$

The dual product of $d\,\Omega$ and a vector field $g = (g_1, \ldots, g_n)^T$ is a scalar field denoted by

$$< d\,\Omega\,, g > = \frac{\partial \Omega}{\partial x_1} g_1 + , \ldots, + \frac{\partial \Omega}{\partial x_n} g_n .$$

With this notion, complete integrability of $\{f_1, \ldots, f_m\}$ can also be deduced from the following fact.

A set of vector fields on R^n, $\{f_1, \ldots, f_m\}$, is completely integrable if and only if there exist n-m linearly independent functions $h_1(x), \ldots, h_{n-m}(x)$ such that

$$<dh_i(x)\,, f_j(x)> = 0 \quad , 1 \leqslant i \leqslant n-m \; , 1 \leqslant j \leqslant m \quad \text{for } all \;\; x \in R^n$$

With the concepts of involutiveness and complete integrability we now state the well-known Frobenius Theorem.

Frobenius Theorem: A set of *linearly independent vector fields* $\{f_1, \ldots, f_m\}$ is completely integrable if and only it is involutive [26].

We are concerned with the class of nonlinear system

$$\dot{x} = f(x) + g(x)u \quad , \quad x \in R^n \; , u \in R$$

which is equivalent to a controllable linear time-invariant system (4.2.1) after external feedback linearization

$$\dot{y} = Ay + Bv \qquad , y \in R^n \; , v \in R \; . \tag{4.2.1}$$

From [27] necessary and sufficient conditions for the local existence of such a transformation are

(i) f(0) = 0.

(ii) the controllability matrix $\left| g \; , ad_f^{\,1}(g) , \ldots, ad_f^{\,n-1}(g) \right|$ span R^n about the origin.

(iii) the set of vector fields $\{ g \; , ad_f^{\,1}(g) , \ldots, ad_f^{\,n-2}(g) \}$ is involutive.

Before we state the conditions for the global existence of such a transformation, we integrate along the involutive distribution $\{ g \ , ad_f^1(g), \ldots, ad_f^{n-2}(g) \}$.

(1) Solve for all $\Omega \in R$ the system

$$\frac{dx}{d\Omega} = ad_f^{n-1}(g) \ , x(0) = 0 \qquad (4.2.2\ a)$$

and obtain the solution $x(\Omega)$.

(2) Solve for all $\theta_1 \in R$ the system

$$\frac{dx}{d\theta_1} = ad_f^{n-2}(g) \ , x(\Omega, 0) = x(\Omega) \qquad (4.2.2.b)$$

and obtain the solution $x(\Omega, \theta_1)$.

(3) Obtain the solution $x(\Omega, \theta_1, \theta_2)$ for all $\theta_2 \in R$ the system

$$\frac{dx}{d\theta_2} = ad_f^{n-3}(g) \ , x(\Omega, \theta_1, 0) = x(\Omega, \theta_1) \qquad (4.2.2.c)$$

(4) Repeat in this manner until we obtain the solution $x(\Omega, \theta_1, \ldots, \theta_{n-1})$ for all $\theta_{n-1} \in R$ the differential equation

$$\frac{dx}{d\theta_{n-1}} = g \ , x(\Omega, \theta_1, \ldots, \theta_{n-2}, 0) = x(\Omega, \theta_1, \ldots, \theta_{n-2}). \qquad (4.2.2.d)$$

Carrying out the above procedure we have the map

$$M: \ x = (x_1, \ldots, x_n) \rightarrow (\Omega, \theta_1, \ldots, \theta_{n-1})$$

which has a Jacobian matrix, or the *noncharacteristic matrix*

$$J(x) = \begin{bmatrix} \frac{\partial x_1}{\partial \Omega} & \frac{\partial x_1}{\partial \theta_1} & \cdots & \frac{\partial x_1}{\partial \theta_{n-1}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial x_n}{\partial \Omega} & \frac{\partial x_n}{\partial \theta_1} & \cdots & \frac{\partial x_n}{\partial \theta_{n-1}} \end{bmatrix}$$

The conditions for the global existence of the inverse of map M is: [28]

there exists a constant $\rho > 0$ such that the absolute values of the leading principal minors

$\Delta_1, \Delta_2, \ldots, \Delta_n$ of J(x) satisfy the *ratio condition*

$$|\Delta_1| \geq \rho, \frac{|\Delta_2|}{|\Delta_1|} \geq \rho, \ldots, \frac{|\Delta_n|}{|\Delta_{n-1}|} \geq \rho$$

for all $x \in R^n$. The scalar Δ_k is defined to be the determinant of the matrix obtained by deleting the last $n-k$ columns and rows of J.

Now we are ready to state the conditions for the global case. Proof can be found in [28].

Theorem 4.2.1

$\dot{x} = f(x) + g(x)u$ is globally transformable to $\dot{y} = Ay + Bv$ with external feedback linearization if

(i) the controllability matrix $\left[g(x), ad_f^1(g(x)), \cdots, ad_f^{n-1}(g(x)) \right]$ is nonsingular on R^n,

(ii) the set $\{g(x), ad_f^1(g(x)), \cdots, ad_f^{n-2}(g(x))\}$ is involutive on R^n, and

(iii) the noncharacteristic matrix satisfies the ratio condition on R^n.

4.3. Problem Formulation

A. System description

Consider the SISO feedback system as shown in Figure 4-1, where P is the given nonlinear plant. Scalars c_1 and δ are, respectively, the plant-output disturbance and plant-input disturbance. The scalar c is the reference input. The variables $\bar{u}$ and y are, respectively, the input and the output of the plant P. The controller F takes E, the error between the output of the plant and that of the reference input plus plant-output disturbance, and produces u that is to be fed into the plant.

The nonlinear plant P with input u, state x, and output y is described by the following equations:

$$\dot{x} = f(x) + g(x)u \tag{4.3.1}$$

$$y = \eta(x) \tag{4.3.2}$$

where $x \in R^n$, $y \in R$, and $u \in R$. The controller is to be designed such that the closed-loop

system performs asymptotic tracking and disturbance rejection *for all given initial conditions* and *for all inputs and disturbances* satisfying our assumptions.

B. General assumptions

The following assumptions are assumed to be satisfied throughout this chapter.

A4.3.1: $f : R^n \rightarrow R^n$ and $g : R^n \rightarrow R^n$ are such that the system $\dot{x} = f(x) + g(x)u$ is globally feedback linearizable.

A4.3.2: $\eta : R^n \rightarrow R$ is a C^2 function .

A4.3.3: The reference input c , the plant-input disturbance δ , and the plant-output disturbance c_1 are all scalar constants (see comments below). We will assume $c_1 = 0$, and hence $E_1 = E$, since its effect can be included in c in the closed-loop system.

A4.3.4: The states of the given plant are available for full-state feedback.

Comments:

From Theorem 4.2.1. Assumption A4.3.1 is required so that our nonlinear system can be transformed into a linear controllable system by diffeomorphism and external feedback linearization. Since f and g are both smooth functions, it therefore guarantees the existence of a unique solution for our plant (4.3.1) for all $t \geq t_0$ with any given initial condition x_0 and initial time t_0 . Homogeneity only serves to ease our discussion and is not necessary. If (x_e , u_e) is the equilibrium of (4.3.1), *i. e.*, $f(x_e) + g(x_e)u_e = 0$, a change of coordinates $\bar{x} = x - x_e$, $\bar{u} = u - u_e$, will bring us back to a homogeneous system. Note also the requirement of $\eta(0) = 0$ as demanded by Desoer and Lin is not required here [25]. For reference input and disturbance varying at a rate of $O(\epsilon^2)$, $0 < \epsilon << 1$, our methodology can still achieve asymptotic tracking and disturbance rejection up to $O(\epsilon)$ neighborhood of the perfect one.

4.4. Asymptotic Tracking of an Unknown Constant Reference Input

Assuming that there is no plant-input disturbance, i. e., $\tilde{u} = u$, we shall show that a nonlinear feedback control plus an integral control will achieve asymptotic tracking of the constant set-point problem.

(1) *External Feedback Linearization*

We seek a change of coordinates for (4.3.1)-(4.3.2):

$$T: R^n \to R^n$$

$$T(x) = z$$

where T is a C^∞ diffeomorphism and in the new local coordinates there exists a function

$$\Omega: R^n \to R$$

such that

$$L_g \Omega(x) = 0$$

$$L_g(L_f \Omega(x)) = 0 \tag{4.4.1}$$

$$\vdots$$

$$L_g(L_f^{n-2}\Omega(x)) = 0$$

$$L_g(L_f^{n-1}\Omega(x)) \neq 0 \tag{4.4.2}$$

where $L_g \Omega$ represents the Lie derivative of Ω along the vector field g.

$$L_g \Omega \equiv < \frac{\partial \Omega}{\partial x}, g >$$

$< \ldots >$ denotes dot product.

It can be shown that the Ω variable we obtained in integrating along the involutive set $\{g(x), ad_f^1(g(x)), \cdots, ad_f^{n-2}(g(x))\}$ as in Section 4.2 satisfies (4.4.1)-(4.4.2) [28]. We shall define the new local coordinates by the following C^∞ diffeomorphism from the old ones.

$$\begin{bmatrix} z_1 \\ z_2 \\ \cdot \\ \cdot \\ \cdot \\ z_n \end{bmatrix} = \begin{bmatrix} \Omega(x) \\ L_f \, \Omega(x) \\ \cdot \\ \cdot \\ \cdot \\ L_f^{n-1}\Omega(x) \end{bmatrix} = \begin{bmatrix} T_1(x) \\ T_2(x) \\ \cdot \\ \cdot \\ \cdot \\ T_n(x) \end{bmatrix} = T(x) \, . \tag{4.4.3}$$

We then have the following equivalent system from (4.3.1):

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= z_3 \\ &\cdot \\ &\cdot \\ &\cdot \\ \dot{z}_{n-1} &= z_n \\ \dot{z}_n &= L_f^n \, \Omega(x) + L_g (L_f^{n-1} \Omega(x)) \, u \\ &= F(x) + G(x) u \, . \end{aligned} \tag{4.4.4}$$

We shall show that a nonlinear feedback control of the form

$$u = G^{-1}(u_1 + u_2 + v) \tag{4.4.5}$$

will achieve asymptotic tracking of the unknown set-point.

Pick $u_1 = -F(x)$, (4.4.4) becomes

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ 0 & 0 & 0 & . & . & . & 0 \\ . & . & . & & & & . \\ . & . & . & & & & . \\ . & . & . & & & & . \\ 0 & 0 & 0 & . & . & . & 1 \\ 0 & 0 & 0 & . & . & . & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 1 \end{bmatrix} (u_2 + v) \tag{4.4.6}$$

which is a standard linear controllable system.

Observe that this external feedback linearizing technique requires full-state feedback and also the nonsingularity of G(x). We can carry out the pole placement design or stabilization of the system once we have our system transformed into (4.4.6).

Hunt, Su, and Meyer had shown that conditions for the global existence of G^{-1} and a C^{∞} transformation (4.4.3) are equivalent to Assumption A4.3.1[28]. Similar argument for feedback linearization can also be found in other books[29, 30].

(2) *Stabilizing the linearized system*

We now apply a feedback control $u_2(z)$ to stabilize the linear system (4.4.6).

$$u_2 = \sum_{i=1}^{n} \alpha_i z_i = \sum_{i=1}^{n} \alpha_i T_i(x) \tag{4.4.7}$$

where α_i 's are chosen so that the resultant system matrix is Hurwitz, i. e., the characteristic polynomial

$$s^n - \alpha_n s^{n-1} - \alpha_{n-1} s^{n-2} - \dots - \alpha_1 = 0 \tag{4.4.8}$$

has all roots with negative real parts.

With u_1 and u_2 the given system becomes

$$\dot{z} = \frac{dz}{d\tau} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} v$$

$$= Az + Bv \,. \tag{4.4.9}$$

We shall study the behavior of the overall system when v is an integral control governed by

$$\frac{dv}{d\tau} = \epsilon \Gamma(E(z), v, \epsilon) \tag{4.4.10.a}$$

$$\Gamma(0, v, \epsilon) = 0 \tag{4.4.10.b}$$

where Γ is a smooth nonlinear function to be designed . E is the tracking error ,and ϵ is a small positive number. The equivalent system is as shown in Figure 4-2.

Remark: When $\epsilon = 0$, $v =$ v(0) becomes a constant. By the nature of Hurwitzness of the system matrix A the states z remain bounded despite the presence of v, which acts as a constant disturbance to the system (4.4.9).

(3) *Change of time-scale and the existence of integral manifold*

We now define $t = \epsilon\tau$ as the slow time scale. In the slow time scale, (4.4.9)-(4.4.10) will be in standard form for the integral manifold discussion as in (1.2.1).

$$\frac{dv}{dt} = \Gamma(E(z), v, \epsilon) \tag{4.4.11}$$

$$\epsilon\frac{dz}{dt} = Az + Bv \tag{4.4.12}$$

It is easy to check that the conditions in Section 1.2 for the existence of the integral manifold for (4.4.11)-(4.4.12) are satisfied.

M1: $Az + Bv = 0$ gives

$$\bar{z}_2 = \bar{z}_3 = \dots = \bar{z}_n = 0 \tag{4.4.13}$$

and

$$\bar{z}_1 = -\bar{v}/\alpha_1 . \tag{4.4.14}$$

Note that in (4.4.14) $\alpha_1 \neq 0$ since if $\alpha_1 = 0$ (4.4.8) will have a zero root contradicting the fact that all of its roots have negative real parts.

M2: Trivial.

M3:

$$\mathrm{Re}\left[\lambda_i(A)\right] < 0, \qquad 1 \leqslant i \leqslant n \quad ,$$

where $\lambda_i(A)$ stands for the eigenvalue of the matrix A.

Trivial by (4.4.8).

Hence, there exists an integral manifold of the form $z = h(v, \epsilon)$ for the system (4.4.11)-(4.4.12).

$$h(v, \epsilon) = h^0(v) + \epsilon h^1(v) + \dots \tag{4.4.15}$$

$h(v, \epsilon)$ can be found by the fact that it satisfies (4.4.11)-(4.4.12), i. e.,

$$\epsilon\frac{\partial h}{\partial v}\dot{v} = Ah(v, \epsilon) + Bv \tag{4.4.16}$$

Using MAE (Matched Asymptotic Expansion), we equate coefficients of successive powers of

ϵ on both sides of (4.4.16)

$$\epsilon^0: \quad 0 = Ah^0(v) + Bv$$

or

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \end{bmatrix} \begin{bmatrix} h_1^0 \\ h_2^0 \\ \cdot \\ \cdot \\ \cdot \\ h_n^0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} v = 0$$

gives

$$h^0(v) = \begin{bmatrix} -v/\alpha_1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} \tag{4.4.17}$$

Coefficients of ϵ^1 terms in (4.4.16):

$$\frac{\partial h^0}{\partial v} \Gamma(E, v, 0)_{|z = h^0(v)} = Ah^1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \end{bmatrix} h^1$$

gives

$$h^1(v) = -\frac{1}{\alpha_1} \Gamma(E, v, 0)_{|z = h^0(v)} \begin{bmatrix} -\alpha_2/\alpha_1 \\ 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

Since E is a function of the plant-output, which in turns depends on the states z, we use $\Gamma(E, v, 0)_{|z=h^0(v)}$ to express the fact that it is evaluated at $z = h^0(v)$. Further analysis with MAE gives $h(v, \epsilon)$ to any order of accuracy in powers of ϵ.

Thus, we have

$$h(v, \epsilon) = h^0(v) + O(\epsilon)$$

$$= \begin{bmatrix} -v/\alpha_1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} + O(\epsilon) \qquad (4.4.18)$$

When the system does not start with its initial condition on the manifold, viz $z(t_0) \neq h(v(t_0), \epsilon)$, there is a deviation from the manifold given by

$$\hat{z} = z - h(v, \epsilon)$$

Lemma 4.4.1

$$\epsilon \frac{dz}{dt} = Az + Bv$$

$$\frac{dv}{dt} = \Gamma(z, v, t, \epsilon)$$

$A \in R^{n \times n}$, $B \in R^n$, $t \in R$, $\Gamma \in C^2$, $\epsilon \in (0, \epsilon^*]$, ϵ^* is a small positive number and A is Hurwitz.

The above system has an integral manifold $z = h(v, t, \epsilon)$ with a global region of attraction for ϵ^* small enough.

Proof: see section 4.8.

With this attractivity property the trajectory asymptotically converges to the integral manifold. Note that we have a *global region of attraction*. When the trajectory is on the manifold the behavior of the system is described by the action of integral control.

The plant-output in the z-coordinates is

$$\eta(x) = \eta(T^{-1}(z)) = w(z)$$

where

$$w \equiv \eta \circ T^{-1}$$

is a C^2 function.

The error between the plant-output and that of the reference input in z-coordinates is

$$E = w(z) - c$$

which, when the system is on the integral manifold, becomes

$$E = w(h(v, \epsilon)) - c .$$

Note that since we are working on a constant set-point problem for the time being, the manifold is of the form $z = h(v, \epsilon)$.

Since $w \in C^2$, we can use the chain rule of differentiation and obtain

$$\frac{dw}{dt} = \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}\frac{dv}{dt} = \frac{\partial w}{\partial z}\frac{\partial h}{\partial v}\frac{dv}{dt}\Big|_{z = h(v, \epsilon)}$$

Now

$$\frac{\partial w}{\partial z}\frac{\partial h}{\partial v} = \frac{\partial w}{\partial z_1}\frac{\partial h_1(v, \epsilon)}{\partial v} + \ldots + \frac{\partial w}{\partial z_n}\frac{\partial h_n(v, \epsilon)}{\partial v}$$

Recall from (4.4.18) we have

$$\frac{\partial h}{\partial v} = \begin{bmatrix} \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial v} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial h_n}{\partial v} \end{bmatrix} = \begin{bmatrix} -1/\alpha_1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix} + O(\epsilon)$$

Therefore,

$$\frac{\partial w}{\partial z}\frac{\partial h}{\partial v} = \frac{-1}{\alpha_1}\frac{\partial w}{\partial z_1}\Big|_{z = (-v/\alpha_1, 0, \ldots, 0)^T} + O(\epsilon) . \tag{4.4.19}$$

For ϵ small enough we further propose the following assumption to assure the nonsingularity

of $\frac{\partial w}{\partial z}\frac{\partial h}{\partial v}$ when the system is on the integral manifold $z = h(v\ ,\epsilon)$.

Assumption A4.4.1:

The choice of diffeomorphism T has the property that $\left|\frac{\partial w}{\partial z_1}\right|_{z=(-v/\alpha_1,0,\ldots,0)^T} = O(1)$ uniformly in v, where $w(z) = \eta(T^{-1}(z)) = \eta(x)$ is the plant-output.

With this assumption and the fact that

$$\frac{\partial w}{\partial z}\frac{\partial h}{\partial v} = \frac{\partial w}{\partial z_1}\frac{\partial h_1(v\ ,\epsilon)}{\partial v} + \ldots + \frac{\partial w}{\partial z_n}\frac{\partial h_n(v\ ,\epsilon)}{\partial v}$$

where

$$\left|\frac{\partial h_1}{\partial v}\right| = |\,1/\alpha_1\,|$$

$$\left|\frac{\partial h_i}{\partial v}\right| = O(\epsilon) \qquad 2 \leqslant i \leqslant n$$

we have

$$\left|\frac{\partial w}{\partial z}\frac{\partial h}{\partial v}\right| = O(1) \qquad \textit{uniformly in } v.$$

This in turns ensures that $w \circ h$ is a bijective mapping. Hence one and only one plant-output will achieve the perfect tracking for each reference input.

Remark: The same assumption is made by Desoer and Lin, who claim the implicit existence of the function h which is *made explicit* by us [25].

Using the update law,

$$\frac{dv}{dt} = -\left[\frac{\partial w}{\partial z}\frac{\partial h}{\partial v}\right]^{-1}(w - c) \qquad (4.4.20)$$

where $\frac{\partial w}{\partial z}$ is understood to be evaluated at $z = h(v\ ,\epsilon)$.

we then have

$$\frac{dw}{dt} = -(w - c) = -E\ . \qquad (4.4.21)$$

Solving

$$w(t) = c + (w(0) - c)e^{-t}$$

Or

$$E(t) = E(0)e^{-t} \ .$$

Therefore, the tracking error goes to zero asymptotically.

Thus, the nonlinear differential equation governing the desired integral control is given by

$$\frac{dv}{dt} = -E \left[\frac{\partial w}{\partial z}\frac{\partial h}{\partial v}\right]^{-1} = E \left[\frac{1}{\alpha_1}\frac{\partial w}{\partial z_1}\right]^{-1} + O(\epsilon) \ . \tag{4.4.22}$$

Or, in the τ time scale,

$$\frac{dv}{d\tau} = \epsilon\alpha_1 \left[\frac{\partial w}{\partial z_1}\right]^{-1} E + O(\epsilon^2) \ . \tag{4.4.23}$$

The assumption on $\dfrac{\partial w}{\partial z_1}$ being uniformly bounded from below preserves the two-time scale property in the overall system.

The overall system will finally converge to the uniformly asymptotic stable equilibrium $(z_e \ , v_e)$ that satisfies

$$0 = E = w(h(v_e \ , \epsilon)) - c \tag{4.4.24.a}$$

$$0 = \epsilon \frac{\partial h}{\partial v}\Gamma(0 \ , v_e \ , \epsilon) = Az_e + Bv_e \ . \tag{4.4.24.b}$$

Solving (4.4.24), we obtain the equilibrium point given by

$$v_e = (w \circ h)^{-1}(c)$$

$$z_e = h(v_e \ , \epsilon) = -A^{-1}Bv_e = \begin{bmatrix} -v_e/\alpha_1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix}$$

We have shown that for *any given initial conditions* the closed-loop system will converge asymptotically to the integral manifold due to its global region of attraction. The motion on

the manifold is governed by (4.4.23), which also converges asymptotically to its equilibrium point with bounded plant states and a zero tracking error. To summarize our result, we have the following theorem.

Theorem 4.4.1:

The nonlinear plant $\dot{x} = f(x) + g(x)u$ with output $y = \eta(x)$ will achieve *asymptotic tracking* of an *unknown constant* set point *for all initial conditions* (x_0, t_0) if a control of the form $u = G^{-1}(u_1 + u_2 + v)$ is applied, where

(a) $G(x) = L_g(L_f^{n-1}\Omega(x))$ $u_1 = -F(x)$, and $F(x) = L_f^n \Omega(x)$ where

$\Omega: R^n \to R$ is a scalar field that satisfies

(i) $\dfrac{\partial \Omega}{\partial x} \left[g(x), ad_f^1 g(x), \ldots, ad_f^{n-2} g(x)\right] = (0, \ldots, 0)$ for *all* $x \in R^n$, and

(ii) $\dfrac{\partial \Omega}{\partial x} ad_f^{n-1} g(x) \neq 0$ for *all* $x \in R^n$

(b) $u_2 = \sum_{i=1}^{n} \alpha_i z_i$ where

$$z = T(x) = \begin{bmatrix} \Omega(x) \\ L_f \Omega(x) \\ \cdot \\ \cdot \\ \cdot \\ L_f^{n-1}\Omega(x) \end{bmatrix}$$

and α_i's are such that for

$$p(s) = s^n - \alpha_n s^{n-1} - \ldots - \alpha_1$$

p(s) has all roots with negative real parts.

(c)

$$\frac{dv}{d\tau} = -\epsilon E \left[\frac{\partial w}{\partial z}\frac{\partial h}{\partial v}\right]^{-1}_{|z = h(v, \epsilon)}$$

$$E = \eta(x) - c \quad , \quad w = \eta \circ T^{-1}$$

$\epsilon \in (0, \epsilon^*]$, ϵ^* is a small positive number that satisfies Lemma 4.4.1 to ensure a globally attrac-

tive manifold. The scalar E is the tracking error.

Remark: With our methodology *exponential stability* as demanded by Desoer and Lin for a constant input u in the system, $\dot{x} = f(x) + g(x)u$, is *not required* [25]. One choice of the scalar field Ω can be obtained by integrating along the involutive set $\{g(x), ad_f^1(g(x)), \cdots, ad_f^{n-2}(g(x))\}$ as in (4.2.2) Section 4.2. Our result here is global. For the local case the assumptions need only be satisfied on the domain of interest instead of the whole R^n space.

With Assumption A4.4.1 on the uniform boundedness of $\left[\frac{\partial w}{\partial z}\frac{\partial h}{\partial v}\right]$ and a small integral gain ϵ, we have the following Corollary.

Corollary 4.4.1

Asymptotic tracking and disturbance rejection for the nonlinear plant $\dot{x} = f(x) + g(x)$ with nonlinear output $y = \eta(x)$ can be achieved if a control $u = G^{-1}(u_1 + u_2 + \hat{v})$ is applied. The functions f, g, G, u_1, u_2, and η are as in Theorem 4.4.1 and

$$\frac{d\hat{v}}{d\tau} = \epsilon\alpha_1 E \left[\frac{\partial w}{\partial z_1}\right]^{-1}_{|z = (-v/\alpha_1, 0, \ldots, 0)^T}$$

where $\epsilon \in (0, \hat{\epsilon}]$, $0 < \hat{\epsilon} << 1$. E is the tracking error.

Proof:

$$\frac{dw}{dt} = \frac{\partial w}{\partial z}\frac{\partial h}{\partial \hat{v}}\frac{d\hat{v}}{dt}_{|z = h(\hat{v}, \epsilon)}$$

With

$$\frac{d\hat{v}}{d\tau} = -\epsilon E \left[\frac{\partial w}{\partial z_1}\frac{\partial h_1^0}{\partial \hat{v}}\right]^{-1}_{|z = (-\frac{\hat{v}}{\alpha_1}, 0, \ldots, 0)^T} = \epsilon\alpha_1 E \left[\frac{\partial w}{\partial z_1}\right]^{-1}_{|z = (-v/\alpha_1, 0, \ldots, 0)^T}$$

and the fact that

$$\frac{\partial w}{\partial z}\frac{\partial h}{\partial \hat{v}} = \frac{\partial w}{\partial z_1}\frac{\partial h_1^0}{\partial \hat{v}} + \epsilon\Psi(\hat{v}, \epsilon)$$

We have

$$\frac{dw}{d\tau} = \frac{\partial w}{\partial z}\frac{\partial z}{\partial \hat{v}}\frac{d\hat{v}}{d\tau} = = -\epsilon E \left|\frac{\partial w}{\partial z}\frac{\partial h}{\partial \hat{v}}\right| \left|\frac{\partial w}{\partial z_1}\frac{\partial h_1^0}{\partial \hat{v}}\right|^{-1}$$

$$= -\epsilon (I + \epsilon\beta(\hat{v}, \epsilon)) E .$$

Recall $E = w - c$, we thus have

$$\frac{dE}{d\tau} = -\epsilon (I + \epsilon\beta(\hat{v}, \epsilon)) E . \tag{4.4.25}$$

With $\beta(\hat{v}, \epsilon)$ being uniformly bounded, there exists a $\tilde{\epsilon}$ such that for all $\epsilon \in (0, \tilde{\epsilon}]$ (4.4.25) is u.a.s.. So with $\hat{\epsilon} = Min\ [\epsilon^{*}, \tilde{\epsilon}]$, $E \to 0$ as $t \to \infty$.

QED

4.5. Disturbance Rejection

We now impose a constant but unknown disturbance δ in our plant-input as shown in Figure 4-1. The actual system is

$$\dot{x} = f(x) + g(x)\bar{u}$$

where

$$\bar{u} = u + \delta .$$

In the new local coordinates described by (4.4.10)

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = z_3$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$\dot{z}_{n-1} = z_n$$

$$\dot{z}_n = F(x) + G(x)\bar{u}$$

where F(x) and G(x) are as defined in (3.11).

When we first apply

$$u_1 = -F(x).$$

it results in

$$\dot{z}_n = G(x)\delta = G(T^{-1}(z))\delta$$

With u_2 and v also taken into consideration, the overall system becomes

$$\frac{dz}{d\tau} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ 1 \end{bmatrix} (v + \delta G(T^{-1}(z))) \tag{4.5.1.a}$$

$$\frac{dv}{d\tau} = \epsilon \Gamma(E, v, \epsilon). \tag{4.5.1.b}$$

We now propose the following assumption.

Assumption A4.5.1: The plant-input disturbance is an unknown but small constant.

With A6 we check the conditions for the existence of an *integral manifold* for (4.5.1).

M1: setting the RHS of (4.5.1.a) to zero gives

$$\bar{z}_2 = \bar{z}_3 = \ldots \bar{z}_n = 0$$

$$\alpha_1 \bar{z}_1 + \bar{v} + \delta G(T^{-1}(\bar{z}_1, \ldots, 0)) = 0 \tag{4.5.2}$$

By the implicit function Theorem for δ small enough (4.5.2) has a unique solution given by

$$\bar{z}_1 = -\bar{v}/\alpha_1 + O(\delta).$$

M2: Trivial.

M3: Taking the partial derivatives with respect to z on the RHS of (4.5.1.a) and evaluate it at $z = h^0(v)$ gives

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_1+\delta_1 & \alpha_2+\delta_2 & \alpha_3+\delta_3 & \cdots & \alpha_n+\delta_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 0 & 0 & 0 & \cdots & 1 \\ \bar{\alpha}_1 & \bar{\alpha}_2 & \bar{\alpha}_3 & \cdots & \bar{\alpha}_n \end{bmatrix} \tag{4.5.3}$$

where

$$\begin{bmatrix} \delta_1 \\ \delta_2 \\ \cdot \\ \cdot \\ \cdot \\ \delta_n \end{bmatrix} = \delta \, \frac{\partial G(T^{-1}(z))}{\partial z} \Big|_{z = h^0(v)}$$

and $h^0(v)$ is as in M1.

The characteristic polynomial of the perturbed system matrix (4.5.3) is given by

$$\bar{P}(s) = s^n - \bar{\alpha}_n s^{n-1} - ,\dots, - \bar{\alpha}_1 . \tag{4.5.4}$$

With Assumption A4.5.1, we now carry out the root sensitivity analysis for (4.5.4).

Suppose P(s) has roots given by

$$P(\lambda_l) = 0 , \quad 1 \leqslant l \leqslant n .$$

i. e.,

$$P(s) = s^n - \alpha_n s^{n-1} - ,\dots, - \alpha_1$$
$$= \prod_{l=1}^{n} (s - \lambda_l) .$$

Denoting $\Delta\lambda_i$ and $\Delta\alpha_k$ as the perturbed part of λ_i and α_k respectively, we have

$$\Delta\lambda_i = \sum_{k=1}^{n} \frac{\partial \lambda_i}{\partial \alpha_k} \Delta\alpha_k . \tag{4.5.5}$$

In our case here

$$\Delta\alpha_k = O(\delta)\,, \quad 1 \leqslant k \leqslant n\,.$$

Now

$$\frac{\partial P}{\partial \alpha_k} = \frac{\partial P}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial \alpha_k}$$

where

$$\frac{\partial P}{\partial \alpha_k} = -s^{k-1} \quad , \quad \frac{\partial P}{\partial \lambda_i} = -\prod_{l=1,\, l \neq i}^{n} (s - \lambda_l)\,.$$

Thus,

$$\frac{\partial \lambda_i}{\partial \alpha_k} = \left[\frac{\partial P}{\partial \lambda_i}\right]^{-1} \left[\frac{\partial P}{\partial \alpha_k}\right]_{|\, s = \lambda_i}$$

$$= \frac{\lambda_i^{k-1}}{\prod_{l=1,\, l \neq i}^{n} (\lambda_i - \lambda_l)} \tag{4.5.6}$$

From (4.5.5)-(4.5.6) it is seen that if we do not have tightly clustered roots, which can be done by eigenvalue placement at our disposal as in (4.4.14), the perturbed system matrix in (4.5.3) will still remain Hurwitz. Hence, the existence of an integral manifold is still guaranteed and the rest of the argument is similar to that in Section 4.4.

Remark: Suppose the plant-input disturbance δ is not restricted to a small perturbation and the integral control is a constant $v(t) \equiv v(0)$, but somehow the closed-loop system remains exponentially stable as assumed by Desoer and Lin [25]. In order that our methodology be feasible in this case, we require the existence of a unique solution $\bar{z}_1(v)$ to (4.5.2) so that the existence of an integral manifold for our system is assured. This is equivalent to the assumptions made by Desoer and Lin for the existence of a C^1 function

$$h:\, R \rightarrow R^n$$

such that

$$\bar{z} = h(\bar{v}\,, \epsilon) \quad \text{for } all \quad \bar{v} \in R$$

where $(\bar{v}\,, \bar{z})$ is the equilibrium point for the closed-loop system and $w \circ h$ is a bijection mapping as deduced from Assumption A4.4.1.

In other words, an alternative assumption for disturbance rejection is the following:

Assumption A4.5.1.a:

(i) The disturbance δ is a bounded constant that gives a bounded solution $z(\tau)$ in (4.5.1) for all constant v and

(ii) $\alpha_1 \bar{z}_1 + \bar{v} + \delta G(T^{-1}(\bar{z}_1, 0, \cdots, 0)) = 0$ has a unique solution $\bar{z}_1 = \bar{z}_1(\bar{v}, \delta)$.

Note for δ small enough the above assumption is always satisfied. With this assumption it can be shown that the existence of an integral manifold $z = h(v, \delta)$ is still satisfied and the rest of the argument is similar to that in Section 4.4.

We have shown how a class of linear equivalent nonlinear system can achieve asymptotic tracking and disturbance rejection for any given initial conditions by external feedback linearization and the integral manifold approach. The nonlinear system $\dot{x} = f(x) + g(x)u$ is first transformed into a stable linear system $\dot{z} = Az + Bv$ by a nonlinear feedback and the use of new local coordinates, $z = T(x)$. An integral control of the form $\dot{v} = \epsilon\Gamma(E, v, \epsilon)$ is adopted. The concept of integral manifold is then used to design the integral control and prove that the plant-output asymptotically tracks the given reference input and simultaneously rejects disturbance.

4.6. Asymptotic Tracking and Disturbance Rejection of Slowly Varying Unknown Signals

We now consider slowly varying unknown bounded signals as our reference input and disturbance. By using the integral manifold concept we shall show how the asymptotic tracking can be achieved when some extra assumptions are satisfied. The overall system is given by

$$\dot{x} = f(x) + g(x)(u + \delta(\tau))$$

$$u = G^{-1}(u_1(x) + u_2(x) + v)$$

$$\dot{v} = \epsilon\Gamma(z, v, \delta, \epsilon, \tau)$$

$$z = T(x)$$

$$E(\tau) = \eta(x) - c(\tau)$$

$$\dot{c} = \epsilon^2 \gamma(\tau)$$

$$\dot{\delta} = \epsilon^2 \sigma(\tau)$$

where $\dot{x} = \dfrac{dx}{d\tau}$ and $\gamma(\tau)$ and $\sigma(\tau)$ are smooth functions that give rise to a bounded reference input c(t), and a small disturbance, i. e.,

Assumption A4.6.1:

The reference input $c(\tau) \in D$, the disturbance $\delta(\tau) \in E$,

$$D = \{ c \in R : |c(\tau)| < B , |\dot{c}(\tau)| = O(\epsilon^2) \text{ , for } all \ \tau \in [\tau_0 , \infty)\}$$

$$E = \{ \delta \in R : |\delta(\tau)| < \hat{\delta} , |\dot{\delta}(\tau)| = O(\epsilon^2) \text{ , for } all \ \tau \in [\tau_0 , \infty)\}$$

where B is a positive number , $\hat{\delta}$ is a small number as presumed in Assumption A4.5.1, and τ_0 is the initial time.

Theorem 4.6.1

For the nonlinear plant $\dot{x} = f(x) + g(x)u$ with nonlinear output $y = \eta(x)$, a slowly varying unknown reference signal $c(\tau)$ and a disturbance $\delta(\tau)$ both satisfying Assumption A4.6.1, the control $u = G^{-1}(u_1 + u_2 + \hat{v})$ will result in tracking error $E(\tau) \to O(\epsilon)$ as $\tau \to \infty$. u_1 , u_2 , and $\hat{v}$ are as in Corollary 4.4.1. $\epsilon \in (0 , \hat{\epsilon}]$ is the integral gain and $0 < \hat{\epsilon} << 1$.

Proof:

In slow time scale $t = \epsilon\tau$

$$\epsilon\frac{dz}{dt} = Az + B(\hat{v} + \delta G) \tag{4.6.1}$$

$$\frac{dv}{dt} = \tilde{\Gamma}(z , \hat{v} , \delta , \epsilon , t) \tag{4.6.2.a}$$

$$\frac{d\delta}{dt} = \epsilon\tilde{\sigma}(t) \tag{4.6.2.b}$$

where $\tilde{\sigma}(t) = \sigma(\tau/\epsilon)$, etc.

By Lemma 4.4.1, (4.6.1)-(4.6.2) has a globally attractive integral manifold

$$z = h(\hat{v} , \delta , \epsilon , t) \tag{4.6.3}$$

and can be solved from the PDE

$$\epsilon\,(\frac{\partial h}{\partial \hat{v}}\,\tilde{\Gamma}(h\ ,\hat{v}\ ,\delta\ ,\epsilon\ ,t\ ) + \epsilon\,\frac{\partial h}{\partial \delta}\tilde{\sigma} + \frac{\partial h}{\partial t}) = Ah\ + B\ (\hat{v}\ + \delta G\ )\ .$$

Denoting

$$h\ = h^{0} + \epsilon\, h^{1} + \epsilon^{2} h^{2} + \ldots\ .$$

we have

$$\epsilon^{0}:\quad Ah^{0} + B\ (\hat{v}\ + \delta G\,(h^{0})) = 0\quad .\ h^{0} = h^{0}(\hat{v}\ ,\delta)$$

$$\epsilon^{1}:\quad \frac{\partial h^{0}}{\partial \hat{v}}\,\tilde{\Gamma}(h^{0}\ ,\hat{v}\ ,\delta\ ,0\ ,t\ ) = Ah^{1} + \frac{\partial}{\partial \epsilon}\left[B\ \delta G\,(h\ )\right]\Big|_{\epsilon=0}\quad .\ h^{1} = h^{1}(\hat{v}\ ,\delta\ ,t\ )\ .$$

etc.

Hence.

$$h\,(\hat{v}\ ,\delta\ ,\epsilon\ ,t\ ) = h^{0}(\hat{v}\ ,\delta) + \epsilon\bar{h}\,(\hat{v}\ ,\delta\ ,\epsilon\ ,t\ )$$

Thus. when the trajectory of (4.6.1)-(4.6.2) is on the manifold (4.6.3). the plant-output becomes

$$w\,(z\ ) = w\,(h\,(\hat{v}\ ,\delta\ ,\epsilon\ ,t\ ))\ .$$

Differentiating the above expression with respect to t gives

$$\frac{dw}{dt} = \frac{\partial w}{\partial z}\,(\frac{\partial h}{\partial \hat{v}}\,\frac{d\hat{v}}{dt} + \epsilon\,\frac{\partial h}{\partial \delta}\,\tilde{\sigma}(t\ ) + \epsilon\frac{\partial \bar{h}}{\partial t}\ ) \tag{4.6.4}$$

$$= (\frac{\partial w}{\partial z_{1}}\,\frac{\partial h_{1}^{0}}{\partial \hat{v}})\,\frac{d\hat{v}}{dt} + \epsilon\ \Psi(\hat{v}\ ,\delta\ ,t\ \ ,\epsilon) = (\frac{-1}{\alpha_{1}})(\frac{\partial w}{\partial z_{1}})\,\frac{d\hat{v}}{dt} + \epsilon\ \Psi(\hat{v}\ ,\delta\ ,t\ \ ,\epsilon)\ .$$

By adopting the integral control

$$\frac{d\hat{v}}{dt} = \alpha_{1}\left[\frac{\partial w}{\partial z_{1}}\right]^{-1} E\ = \alpha_{1}\left[\frac{\partial w}{\partial z_{1}}\right]^{-1}\ (w\ - c\ )\ . \tag{4.6.5}$$

we have from (4.6.4)

$$\frac{dw}{dt} = -(w\ - c\ ) + \epsilon\Psi\ . \tag{4.6.6}$$

Now by using a new relative slow time scale $s\ = \epsilon t$, we have the following subsystem:

$$\epsilon \frac{dw}{ds} = -(w - c) + \epsilon \Psi \tag{4.6.7.a}$$

$$\frac{dc}{ds} = \bar{\gamma}(s) \tag{4.6.7.b}$$

where $\bar{\gamma}(s) = \gamma(\tau/\epsilon^2)$. It is easy to see that (4.6.7) has the following integral manifold:

$$w = W(c , s , \epsilon) \tag{4.6.8}$$

where W satisfies the following PDE

$$\epsilon\left(\frac{\partial W}{\partial c}\bar{\gamma} + \frac{\partial W}{\partial s}\right) = -(W - c) + \epsilon \Psi .$$

Solving

$$W(c , s , \epsilon) = c - \epsilon(\bar{\gamma}(s) + \Psi_0) + \cdots \tag{4.6.9}$$

Overall, the trajectory will asymptotically converge to the integral manifold (4.6.3) and then later to the integral manifold (4.6.9) within the manifold (4.6.3).

Note when the subsystem (4.6.7) is on the manifold (4.6.8) we have from (4.6.9) the following:

$$w - c = O(\epsilon) . \tag{4.6.10}$$

By (4.6.10) and (4.6.6) we conclude that asymptotic tracking and disturbance rejection to $O(\epsilon)$ neighborhood of origin can be achieved when an integral control of the form (4.6.4) is used.

QED

4.7. Example and Simulations

Consider the nonlinear system on R^2

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 + \frac{x_1^2}{2} + x_2 + e^{x_2} - 1 \\ x_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{4.7.1}$$

$$= f(x) + g(x) u$$

where u represents control and $\dot{x} = \dfrac{dx_1}{d\tau}$ is the time derivative of x_1 etc. Note that (4.7.1) has an unstable equilibrium at the origin because

$$\frac{\partial f}{\partial x}\Big|_{(0,0)} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

Compute the *Lie bracket*,

$$\left[f \ , g \right] = \begin{bmatrix} -(1 + e^{x_2}) \\ 0 \end{bmatrix},$$

it is easy to show that $\{ g \ , \left[f \ , g \right] \}$ is linearly independent on R^2.

Integrating along $\{ g \ , \left[f \ , g \right] \}$ gives

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2\Omega \\ \theta \end{bmatrix}. \tag{4.7.2}$$

The noncharacteristic matrix of (4.7.2) is

$$J = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

which satisfies the ratio condition with $\rho = 1/2$ [28].

Thus, we are assured that (4.7.1) is external feedback linearizable on R^2.

For our new local coordinates we choose

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \Omega(x) \\ L_f \ \Omega(x) \end{bmatrix} = \begin{bmatrix} -x_1/2 \\ \dfrac{-1}{2}(x_1 + \dfrac{x_1^2}{2} + x_2 + e^{x_2} - 1) \end{bmatrix},$$

differentiated with respect to τ gives

$$\frac{dz_1}{d\tau} = z_2$$

$$\frac{dz_2}{d\tau} = F(x) + G(x)u$$

where

$$F(x) = \frac{-1}{2}\left((1 + x_1)(x_1 + \frac{x_1^2}{2} + x_2 + e^{x_2} - 1) + x_1^2(1 + e^{x_2}) \right)$$

$$G(x) = -\frac{1}{2}(1 + e^{x_2}) .$$

It is easy to see that $G^{-1}(x)$ exists on R^2.

Following a discussion in Section 4.4 we now apply a nonlinear feedback of the form

$$u = G^{-1}(u_1 + u_2 + v)$$

where u_1 is used to cancel the nonlinearity and is given by

$$u_1 = -F(x) .$$

The linearized system is stabilized by u_2. A choice of u_2 is

$$\begin{aligned} u_2 &= -6z_1 - 5z_2 \\ &= 3x_1 + 2.5(x_1 + \frac{x_1^2}{2} + e^{x_2} + x_2 - 1) . \end{aligned}$$

The equivalent stable linear system is then given by

$$\begin{bmatrix} \dfrac{dz_1}{d\tau} \\ \dfrac{dz_2}{d\tau} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v . \tag{4.7.3}$$

Suppose our plant output is

$$\begin{aligned} y &= \eta(x) = e^{x_1} + x_1 \\ &= w(z) = e^{-2z_1} - 2z_1 . \end{aligned}$$

Let the unknown set-point be denoted by c.

The tracking error in z coordinates is

$$E(t) = -2z_1 + e^{-2z_1} - c = w(z) - c .$$

Equation (4.7.3) in slow time scale $t = \epsilon\tau$ is

$$\begin{bmatrix} \epsilon\dfrac{dz_1}{dt} \\ \epsilon\dfrac{dz_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v \tag{4.7.4}$$

which together with nonlinear integral control

$$\frac{dv}{dt} = \Gamma(E(z), v, \epsilon)$$

have an integral manifold $z = h(v, \epsilon) = h^0(v) + \epsilon h^1(v) + \cdots$ and h satisfies

$$\epsilon \frac{\partial h}{\partial v} \dot{v} = Ah(v, \epsilon) + Bv .$$

The M_0 manifold is obtained by solving

$$0 = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} h^0 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v$$

So

$$h^0(v) = -\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} v = \begin{bmatrix} 1/6 \\ 0 \end{bmatrix} v$$

So

$$\left[\frac{\partial w}{\partial z}\frac{\partial h^0}{\partial v}\right] = \left[-2(1 + e^{-2z_1}) \;\; 0\right] \begin{bmatrix} 1/6 \\ 0 \end{bmatrix} = -\frac{1}{3}(1 + e^{-2z_1})$$

When the nonlinear integral control is governed by

$$\frac{dv}{d\tau} = -\epsilon \left[\frac{\partial w}{\partial z}\frac{\partial h^0}{\partial v}\right]^{-1} E = \frac{-\epsilon E(t)}{-\frac{1}{3}(1 + e^{-2z_1})} = \frac{3\epsilon E(t)}{1 + e^{x_1}}$$

we have

$$\frac{dw}{d\tau} = \left[\frac{\partial w}{\partial z}\frac{\partial h}{\partial v}\right] \frac{dv}{d\tau} = -\epsilon(1 + O(\epsilon))E = -\epsilon(1 + O(\epsilon))(w - c) .$$

Consequently, $E(t) \to 0$ asymptotically. A simulation with set-point $c = 1 + e^1$, $\epsilon = 0.05$ and various initial conditions is shown in Figure 4-3. It is worth remarking that in a constant set-point problem perfect asymptotic tracking is achieved even though we design our integral controller based on $O(\epsilon)$ approximated manifold. Regardless of the initial conditions, all trajectories converge to the manifold and then flow along it towards the equilibrium point. As can be seen from Figure 4-3, the z_1 - coordinate of the equilibrium point is -1/2, which corresponds to $x_1 = 1$. As expected, all the trajectories asymptotically converge to the equilibrium point where we achieve perfect tracking. Figure 4-4 shows the tracking error E(t) asymptotically goes to zero. When an additional unknown bounded disturbance is also added,

the simulation in Figure 4-5 shows that asymptotic tracking is still achieved. As shown in Figure 4-6, with a new value of $\epsilon = 0.018$, $O(\epsilon)$ asymptotic tracking is still achieved when both reference input and disturbance are slowly varying. $\delta = 3.718 - 0.0972\, sin\,(\tau/300)$ and $c = 0.5 - 1.62 \times 10^{-2} cos\,(\tau/50)$.

4.8. Proof of Lemma 4.4.1

Proof:

In order that $h(v, t, \epsilon)$ be an integral manifold for the system, it is necessary that it satisfies the following PDE:

$$\epsilon\left(\frac{\partial h}{\partial v}\Gamma(h(v, t, \epsilon), v, t, \epsilon) + \frac{\partial h}{\partial t}\right) = Ah(v, t, \epsilon) + Bv \;. \tag{4.8.1.a}$$

Let

$$h(v, t, \epsilon) = h^0(v, t) + \epsilon\tilde{h}(v, t, \epsilon) \;. \tag{4.8.1.b}$$

Equating the coefficients of powers of ϵ^0 on both sides of (4.8.1.a),

$$\epsilon^0 \colon \; Ah^0(v, t) + Bv = 0 \tag{4.8.2}$$

or

$$h^0(v, t) = h^0(v, t) = h^0(v) = -A^{-1}Bv$$

with (4.8.1.b)

$$\frac{\partial h}{\partial v}\Gamma(h(v, t, \epsilon), v, t, \epsilon) + \frac{\partial h}{\partial t} = A\tilde{h}(v, t, \epsilon) \tag{4.8.3}$$

Since A is Hurwitz, there exists a positive definite symmetric matrix P such that

$$A^T P + PA < -C$$

where $C > 0$.

To show that the integral manifold has a global region of attraction we shall prove that the deviation from the manifold goes to zero asymptotically for all initial condition $(\hat{z}_0, t_0)$ with.

$$\hat{z} \equiv z - h(v, t, \epsilon)$$

Taking derivatives with respect to t on both sides of the above expression gives

$$\begin{aligned}\epsilon\frac{d\hat{z}}{dt} &= \epsilon\frac{dz}{dt} - \epsilon\left(\frac{\partial h}{\partial v}\frac{dv}{dt} + \frac{\partial h}{\partial t}\right) \\ &= Az + Bv - \epsilon\left(\frac{\partial h}{\partial v}\Gamma(z, v, t, \epsilon)) + \frac{\partial h}{\partial t}\right)\end{aligned}$$

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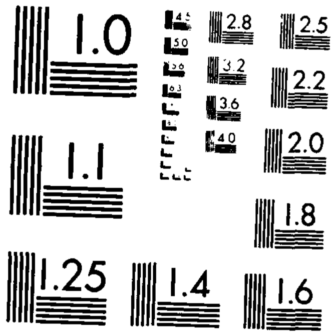

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$$= A(\hat{z} + h(v, t, \epsilon)) + Bv - \epsilon\left(\frac{\partial h}{\partial v}\Gamma(\hat{z} + h(v, t, \epsilon), v, t, \epsilon) + \frac{\partial h}{\partial t}\right)$$

$$= A(\hat{z} + h^0(v)) + Bv + \epsilon\left(A\tilde{h} - \frac{\partial h}{\partial v}\Gamma(\hat{z} + h(v, t, \epsilon), v, t, \epsilon) - \frac{\partial h}{\partial t}\right).$$

By (4.8.2) and (4.8.3)

$$\epsilon\frac{d\hat{z}}{dt} = A\hat{z} + \epsilon\frac{\partial h}{\partial v}\left(\Gamma(h(v, t, \epsilon), v, t, \epsilon) - \Gamma(\hat{z} + h(v, t, \epsilon), v, t, \epsilon)\right).$$

We now use the Lyapunov function $v(\hat{z}) = \hat{z}^T P\hat{z} > 0$ to show that the above equation has an asymptotically stable equilibrium at the origin.

$$\epsilon\dot{v} < -\hat{z}^T C\hat{z} + 2\epsilon\hat{z}^T P\frac{\partial h}{\partial v}\left(\Gamma(h(v, t, \epsilon), v, t, \epsilon) - \Gamma(\hat{z} + h(v, t, \epsilon), v, t, \epsilon)\right)$$

Since $\Gamma \in C^2$, Γ is Lipschitzian, i. e.,

$$\left|\Gamma(h(v, t, \epsilon), v, t, \epsilon) - \Gamma(\hat{z} + h(v, t, \epsilon), v, t, \epsilon)\right| < L|\hat{z}|$$

where L is a positive constant.

Also, since

$$\frac{\partial h}{\partial v} = \frac{\partial h^0}{\partial v} + \epsilon\frac{\partial \tilde{h}}{\partial v} = -A^{-1}B + \epsilon\frac{\partial \tilde{h}}{\partial v}.$$

we have

$$\left\|\frac{\partial h}{\partial v}\right\| < 2\|A^{-1}B\|.$$

Pick $k_1, k_2 > 0$ such that $\hat{z}^T C\hat{z} \geq k_1|\hat{z}|^2$ and $\hat{z}^T P\hat{z} \geq k_2|\hat{z}|^2$ for all $\hat{z} \in R^n$. With $\epsilon \in (0, \epsilon^*]$ where $\epsilon^* = k_1/5k_2L\ \|A^{-1}B\|$ we have $\dot{v} < 0$ uniformly in $\hat{z}$ and hence the uniform asymptotic stability of the system governing the deviation from the integral manifold.

QED

5. OPTIMAL CONTROL SYSTEMS

5.1. Introduction

We have shown in Section 2.2 how a linear system with slow and fast modes

$$\begin{bmatrix} \dot{x} \\ \epsilon\dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \quad , \ A_{22} \ \textit{nonsingular}$$

is equivalent to a pure slow problem with a lower state dimension

$$\dot{x} = (a_{11} + a_{12} L) x$$

$$z(t) = L\, x(t)$$

provided that the initial conditions are on the integral manifold $z = Lx$, i. e., $z(t_0) = Lx(t_0)$ and L satisfies (2.2.3).

This type of model reduction is made possible when the initial conditions are restricted to the manifold. We shal' see in Section 5.2 that there exists a reduced order optimal linear-quadratic system which is equivalent to a higher order regulation problem with slow and fast dynamics. When the existence of an integral manifold is assured in the optimal system, we then pursue the problem where the initial conditions do not start on the manifold. It will be shown that the optimal system can be decomposed into two subsystems. One of these is a decoupled optimal subsystem that governs the convergence of the trajectory to the slow manifold. For the system with slow(O(1)) and fast($O(1/\epsilon)$) modes, the cost required to bring the trajectory to the manifold is of $O(\epsilon)$ in the overall optimal cost. A complete decomposition of the optimal system into decoupled pure slow and pure fast subsystems characterizes the slow-fast behavior of the optimal trajectory. For fixed end-point tracking problems we propose an approximate scheme that renders similar analysis applicable.

5.2. Linear-quadratic Optimal Problems as Restricted to the Integral Manifold

We study the regulation problem of the following system:

$$\begin{bmatrix} \dot{x} \\ \epsilon \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \quad , \; A_{22} \; nonsingular \; . \; x \in R^n \; . \; z \in R^m \; , \; u \in R^r \qquad (5.2.1)$$

$$J = \frac{1}{2} \int_0^\infty x' Qx + z' Sz + u' Ru \; dt \qquad (5.2.2)$$

$$\begin{bmatrix} x \\ z \end{bmatrix}_{t=0} = \begin{bmatrix} x^0 \\ z^0 \end{bmatrix} , \qquad \begin{bmatrix} x \\ z \end{bmatrix}_{t=\infty} \; free$$

where A_{ij}'s and B_i's are understood to be of appropriate dimensions. Matrix R is positive definite and Q and S are both positive definite or positive semi-definite matrices. The constant ϵ is a small positive number. Vector V' stands for the transpose of the vector V.

We are seeking an optimal control, u^*, that minimizes the scalar cost functional, J. Consider the reduced problem, i. e., $\epsilon = 0$. From (5.2.1)

$$\bar{z} = -A_{22}^{-1} \left(A_{21}\bar{x} + B_2\bar{u} \right) \qquad (5.2.3)$$

Substituting this into (5.2.1)-(5.2.2) we have the following reduced linear-quadratic problem:

$$\dot{\bar{x}} = A_0 \bar{x} + B_0 \bar{u}$$

$$\underset{\bar{u}}{Min} \; J_0 = \underset{\bar{u}}{Min} \; \frac{1}{2} \int_0^\infty \bar{x}' Q_0 \bar{x} + 2\bar{u}' C_0 \bar{x} + \bar{u}' R_0 \bar{u} \; dt$$

$$\bar{x}(0) = x^0 \; , \; \bar{x}(\infty) \; free$$

where

$$A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}$$

$$B_0 = B_1 - A_{12}A_{22}^{-1}B_2$$

$$Q_0 = Q + (A_{22}^{-1}A_{21})' S (A_{22}^{-1}A_{21})$$

$$R_0 = R + (A_{22}^{-1}B_2)' S (A_{22}^{-1}B_2)$$

$$C_0 = (A_{22}^{-1}B_2)' S (A_{22}^{-1}A_{21})$$

It is obvious that the reduced optimal problem is easier to solve, due to its lower state dimension. Note that a coupling term $\bar{u}' C_0 \bar{x}$ appears in the reduced cost functional J_r. To facilitate our discussion, we adopt the following assumption:

Assumption 5.2.1

The reduced problem has a unique optimal solution.

Derive the optimality condition for the reduced system by Hamiltonian formulation[31]

$$H = \frac{1}{2}(\bar{x}'Q_0\bar{x} + 2\bar{u}'C_0\bar{x} + \bar{u}'R_0\bar{u}) + \bar{\lambda}'(A_0\bar{x} + B_0\bar{u})$$

where $\bar{\lambda}$ is the costate variable and satisfies the following state equation:

$$\dot{\bar{\lambda}} = -\frac{\partial H_0}{\partial \bar{x}} = -Q_0\bar{x} - A_0'\bar{\lambda} - C_0\bar{u}^* .$$

In order that $\bar{u}^*$ be an optimal control it is necessary that

$$\frac{\partial H_0}{\partial \bar{u}} = R_0\bar{u}^* + B_0'\bar{\lambda} + C_0\bar{x} = 0 .$$

Thus,

$$\bar{u}^* = -R_0^{-1}(B_0'\bar{\lambda} + C_0\bar{x}) .$$

Thus, the optimality conditions for the reduced system are

$$\begin{bmatrix} \dot{\bar{x}} \\ \dot{\bar{\lambda}} \end{bmatrix} = \begin{bmatrix} A_0 - B_0R_0^{-1}C_0 & -B_0R_0^{-1}B_0' \\ -Q_0 + C_0R_0^{-1}C_0 & -(A_0 - B_0R_0^{-1}C_0)' \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{\lambda} \end{bmatrix} \tag{5.2.4}$$

$$\bar{x}(0) = x^0 \quad , \quad \bar{\lambda}(\infty) = 0 .$$

When ϵ is small but nonzero we can use the manifold idea to obtain an equivalent system. Since we have a system (5.2.1) subject to the constraint of minimizing (5.2.2), it is not easy to see the existence of the integral manifold. Instead we investigate the closed-loop system when the optimality condition is obtained from the associate Hamiltonian equation.

$$H = \frac{1}{2}(x'Qx + z'Sz + u'Ru) + \lambda_x'(A_{11}x + A_{12}z + B_1u) + \frac{\tilde{\lambda}_z'}{\epsilon}(A_{21}x + A_{22}z + B_2u)$$

$$= \frac{1}{2}(x'Qx + z'Sz + u'Ru) + \lambda_x'(A_{11}x + A_{12}z + B_1u) + \lambda_z'(A_{21}x + A_{22}z + B_2u)$$

where λ_x and $\tilde{\lambda}_z$ are the associate costate variables. The variable $\lambda_z = \frac{\tilde{\lambda}_z}{\epsilon}$ is the scaled costate variable. In order that u^* be a minimizing control to (5.2.1)-(5.2.2), it is necessary

that,

$$\frac{\partial H}{\partial u} = 0 \quad \text{and} \quad \frac{\partial^2 H}{\partial u^2} > 0 ,$$

i. e.,

$$Ru^* + B_1' \lambda_x + B_2' \lambda_z = 0 \tag{5.2.5}$$

and

$$R > 0 . \tag{5.2.6.a}$$

Equation (5.2.6.a) is trivially satisfied since we assume that R is a positive definite matrix. (5.2.5) gives

$$u^* = -R^{-1} (B_1' \lambda_x + B_2' \lambda_z) \tag{5.2.6.b}$$

The standard calculus of variation approach to optimal problems[31] yields the following costate equations:

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = -Qx - A_{11}' \lambda_x - A_{21}' \lambda_z$$

$$\epsilon \dot{\lambda}_z = -\frac{\partial H}{\partial z} = -Sz - A_{12}' \lambda_x - A_{22}' \lambda_z$$

We then come to the optimality conditions:

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda}_x \\ \epsilon \dot{z} \\ \epsilon \dot{\lambda}_z \end{bmatrix} = \begin{bmatrix} A_{11} & -B_1 R^{-1} B_1' & A_{12} & -B_1 R^{-1} B_2' \\ -Q & -A_{11}' & 0 & -A_{21}' \\ A_{21} & -B_2 R^{-1} B_1' & A_{22} & -B_2 R^{-1} B_2' \\ 0 & -A_{12}' & -S & -A_{22}' \end{bmatrix} \begin{bmatrix} x \\ \lambda_x \\ z \\ \lambda_z \end{bmatrix} \tag{5.2.7}$$

$$\begin{bmatrix} x \\ z \end{bmatrix}_{t=0} = \begin{bmatrix} x^0 \\ z^0 \end{bmatrix}, \qquad \begin{bmatrix} \lambda_x \\ \lambda_z \end{bmatrix}_{t=\infty} = 0$$

Since this is an infinite-time linear-quadratic regulation problem, we have the equivalent end conditions

$$\begin{bmatrix} x \\ z \end{bmatrix}_{t=0} = \begin{bmatrix} x_0 \\ z_0 \end{bmatrix}, \qquad \begin{bmatrix} x \\ z \end{bmatrix}_{t=\infty} = 0$$

In view of the cost functional to be minimized, a good controller should drive the states to zero

as time tends to infinity. We now show that the reduced optimality condition obtained by setting $\epsilon = 0$ in (5.2.7) is identical to the optimality conditions of the reduced optimal problem (5.2.1)-(5.2.2).

Rewriting the optimality conditions (5.2.7) by keeping u^* and setting $\epsilon = 0$ gives

$$\dot{x} = A_{11}x + A_{12}z^0 + B_1u^* \tag{5.2.8}$$

$$0 = A_{21}x + A_{22}z^0 + B_2u^* \tag{5.2.9}$$

$$\dot{\lambda}_x = -Qx - A_{11}'\lambda_x - A_{21}'\lambda_z^0 \tag{5.2.10}$$

$$0 = -Sz^0 - A_{12}'\lambda_x - A_{22}'\lambda_z^0 \tag{5.2.11}$$

where u^* is given by

$$\frac{\partial H}{\partial u} = Ru^* + B_1'\lambda_x + B_2'\lambda_z^0 = 0$$

Eliminating λ_z^0 by (5.2.11), we have the reduced optimality conditions

$$\dot{x} = A_{11}x + A_{12}z^0 + B_1u^* \tag{5.2.12}$$

$$\dot{\lambda}_x = -Qx - A_{11}'\lambda_x + A_{21}'A_{22}'^{-1}(Sz^0 + A_{12}'\lambda_x) \tag{5.2.13}$$

$$x(0) = x^0 \quad , \quad x(\infty) = 0$$

where u^* satisfies

$$Ru^* + B_1'\lambda_x - B_2'A_{22}'^{-1}(Sz^0 + A_{12}'\lambda_x) = 0 \tag{5.2.14}$$

and z^0 satisfies

$$0 = A_{21}x + A_{22}z^0 + B_2u^* \tag{5.2.15}$$

Equations (5.2.12)-(5.2.15) are the reduced optimality conditions.

We now study the reduced system and find its optimality conditions. When we set $\epsilon = 0$ in (5.2.1)-(5.2.2), the optimal infinite-time regulator problem becomes

$$\dot{x} = A_{11}x + A_{12}\tilde{z} + B_1u \tag{5.2.16.a}$$

$$0 = A_{21}x + A_{22}\tilde{z} + B_2u \tag{5.2.16.b}$$

The associate Hamiltonian equation is

$$H = \frac{1}{2}(x'Qx + \tilde{z}'S\tilde{z} + u'Ru) + \lambda_x'(A_{11}x + A_{12}\tilde{z} + B_1u) \tag{5.2.16.c}$$

where λ_x is the associate costate variable and $\tilde{z}(x, u)$ satisfies (5.2.16.b).

Necessary conditions for optimality are

$$\frac{\partial H}{\partial u} = Ru^* + B_1'\lambda_x + \frac{\partial \tilde{z}}{\partial u}(S\tilde{z} + A_{12}'\lambda_x) = 0$$

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = -Qx - A_{11}'\lambda_x - \frac{\partial \tilde{z}}{\partial x}(S\tilde{z} + A_{12}'\lambda_x)$$

From (5.2.16.b)

$$\frac{\partial \tilde{z}}{\partial u} = -B_2'A_{22}'^{-1} \quad \text{and} \quad \frac{\partial \tilde{z}}{\partial x} = -A_{21}'A_{22}'^{-1}$$

So the reduced optimality condition for the reduced optimal problem is

$$\dot{x} = A_{11}x + A_{12}\tilde{z} + B_1u \tag{5.2.17}$$

$$\dot{\lambda}_x = -Qx - A_{11}'\lambda_x + A_{21}'A_{22}'^{-1}(S\tilde{z} + A_{12}'\lambda_x) \tag{5.2.18}$$

$$x(0) = x^0 \quad , \quad x(\infty) = 0$$

u^* satisfies

$$Ru^* + B_1'\lambda_x - B_2'A_{22}'^{-1}(S\tilde{z} + A_{12}'\lambda_x) = 0 \tag{5.2.19}$$

and $\tilde{z}$ satisfies

$$0 = A_{21}x + A_{22}\tilde{z} + B_2u^* \tag{5.2.20}$$

Comparing (5.2.17)-(5.2.20) with (5.2.12)-(5.2.15) we come to the following Lemma:

Lemma 5.2.1

The reduced problem (5.2.16) is formally correct.

Now, we recapitulate what we have done. We derived the necessary optimality conditions for the full optimal system (5.2.1)-(5.2.2) and then obtained the reduced optimality conditions (5.2.8)-(5.2.11) by setting $\epsilon = 0$ in (5.2.7). They were compared with the optimality conditions of the reduced optimal problem (5.2.16) and were found to be the same. In other words, the reduced optimality conditions (5.2.8)-(5.2.11) correspond to the optimal problem given by (5.2.16). Assumption 5.2.1 implies that a unique solution to (5.2.19) exists. The nonsingularity assumption on A_{22} made the above discussion possible.

Observe that the optimality conditions (5.2.7) can be viewed as an initial value problem with

$$\begin{bmatrix} \lambda_x \\ \lambda_z \end{bmatrix}_{t=0} = K \begin{bmatrix} x \\ z \end{bmatrix}_{t=0}$$

where K satisfies an algebraic Riccati equation associated with the entries of the system matrix in (5.2.7) [31]. Thus, we can rewrite (5.2.7) as

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda}_x \\ \epsilon\dot{z} \\ \epsilon\dot{\lambda}_z \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} x \\ \lambda_x \\ z \\ \lambda_z \end{bmatrix} = \Gamma \begin{bmatrix} x \\ \lambda_x \\ z \\ \lambda_z \end{bmatrix} \quad , \quad \begin{bmatrix} x \\ \lambda_x \\ z \\ \lambda_z \end{bmatrix}_{t=0} = \begin{bmatrix} x^0 \\ \lambda_x^0 \\ z^0 \\ \lambda_z^0 \end{bmatrix}$$

where $F_{11} \in R^{2n \times 2n}$, $F_{12} \in R^{2n \times 2m}$, $F_{21} \in R^{2m \times 2n}$, and $F_{22} \in R^{2m \times 2m}$ correspond to appropriate entries of the system matrix in (5.2.7). It is known that Γ has $(2n + 2m)$ eigenvalues of which 2n are slow (O(1)) and 2m are fast $(O(1/\epsilon))$. From[32] it was pointed out that half of the eigenvalues of Γ are symmetric to the other half with respect to the origin. Because of the clear slow-fast separation due to the smallness of ϵ, we can conclude that Γ has n slow eigenvalues which are symmetric to the other n slow eigenvalues and the same for its 2m fast eigenvalues. With these facts we are ready to prove the main theorem with the following assumption:

Assumption 5.2.2

F_{22} is nonsingular.

Theorem 5.2.1

There exists a lower order optimal problem

$$\dot{x}_\epsilon = A_\epsilon x_\epsilon + B_\epsilon u_\epsilon \quad x_\epsilon \in R^n \ , u_\epsilon \in R^r \tag{5.2.21.a}$$

$$\underset{u_\epsilon}{Min}\, J_\epsilon = \underset{u_\epsilon}{Min}\, \frac{1}{2} \int_0^\infty x_\epsilon' Q_\epsilon x_\epsilon + 2u_\epsilon' C_\epsilon x_\epsilon + u_\epsilon' R_\epsilon u_\epsilon \, dt \qquad (5.2.21.b)$$

$$x_\epsilon(0) = x_\epsilon^0 = x^0 \;,\; x_\epsilon(\infty) \;\; free$$

that is equivalent to the optimal regulator problem (5.2.1)-(5.2.2) for some initial conditions $(x^0, z^0)^T$.

This means that the optimal controls in two optimal systems are the same:

$$u^*(t) = u_\epsilon^*(t) \quad , t \geqslant 0$$

Furthermore, the state trajectories along this optimal control are also the same:

$$x^*(t) = x_\epsilon^*(t) \quad , t \geqslant 0$$

Proof:

The optimality conditions of the full-order optimal problem (5.2.1)-(5.2.2) is in the standard form of a singularly perturbed linear system

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda}_x \\ \epsilon\dot{z} \\ \epsilon\dot{\lambda}_z \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} x \\ \lambda_x \\ z \\ \lambda_z \end{bmatrix} , \quad \begin{bmatrix} x \\ \lambda_x \\ z \\ \lambda_z \end{bmatrix}_{t=0} = \begin{bmatrix} x^0 \\ \lambda_x^0 \\ z^0 \\ \lambda_z^0 \end{bmatrix} \qquad (5.2.22)$$

$$u^* = -R^{-1}(B_1'\lambda_x + B_2'\lambda_z) .$$

By Corollary 2.2.1, there exists an integral manifold (5.2.23) in (5.2.22).

$$\begin{bmatrix} z \\ \lambda_z \end{bmatrix} = L \begin{bmatrix} x \\ \lambda_x \end{bmatrix} \qquad (5.2.23)$$

where L $\in R^{2m \times 2n}$ satisfies

$$F_{21} + F_{22}L = \epsilon L(F_{11} + F_{12}L) .$$

Furthermore,

$$L(\epsilon) = -F_{22}^{-1}F_{21} + O(\epsilon) .$$

Thus, (5.2.22) is equivalent to the following lower order system (5.2.24)-(5.2.25), provided

$$\begin{bmatrix} z^0 \\ \lambda_z^0 \end{bmatrix} = L \begin{bmatrix} x^0 \\ \lambda_x^0 \end{bmatrix} .$$

i. e., the initial conditions are on the integral manifold (5.2.23).

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda}_x \end{bmatrix} = (F_{11} + F_{12}L(\epsilon)) \begin{bmatrix} x \\ \lambda_x \end{bmatrix} = \Phi_\epsilon \begin{bmatrix} x \\ \lambda_x \end{bmatrix} \quad , \quad \begin{bmatrix} x \\ \lambda_x \end{bmatrix}_{t=0} = \begin{bmatrix} x^0 \\ \lambda_x^0 \end{bmatrix} \tag{5.2.24}$$

$$\begin{bmatrix} z(t) \\ \lambda_z(t) \end{bmatrix} = L \begin{bmatrix} x(t) \\ \lambda_x(t) \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda_x(t) \end{bmatrix} \quad t \geqslant 0 \tag{5.2.25}$$

and the optimal control u^* on the manifold becomes

$$u^* = -R^{-1}(B_2' L_{21} x + (B_1' + B_2' L_{22})\lambda_x)) \tag{5.2.26}$$

Note the dependence of L on the parameter ϵ. At $\epsilon = 0$ we have the reduced optimality conditions from (5.2.22), and by Lemma 5.2.1 this is identical to the optimality conditions (5.2.4) of the reduced system. Thus, we have

$$(F_{11} + F_{12}L_0) = \Phi_0 = \begin{bmatrix} A_0 - B_0 R_0^{-1} C_0 & -B_0 R_0^{-1} B_0' \\ -Q + C_0 R_0^{-1} C_0 & -(A_0 - B_0 R_0^{-1} C_0)' \end{bmatrix} \tag{5.2.27}$$

Note that $(\Phi_0)_{22} = -(\Phi_0)_{11}'$.

For $\epsilon \neq 0$

$$\Phi_\epsilon = \Phi_0 + \epsilon\, \Phi_1(\epsilon)$$

To show that $(\Phi_\epsilon)_{22} = -(\Phi_\epsilon)_{11}'$. We shall prove it by contradiction. Suppose $(\Phi_\epsilon)_{22} \neq -(\Phi_\epsilon)_{11}'$. We understand that Φ_ϵ has 2n eigenvalues and half of these are symmetric to the other half with respect to the origin. So with some elementary row operations applied to Φ_ϵ, we can obtain a matrix $\hat{\Phi}_\epsilon$ with $(\hat{\Phi}_\epsilon)_{22} = -(\hat{\Phi}_\epsilon)_{11}'$.

$$\hat{\Phi}_\epsilon = E\, \Phi_\epsilon = E\, \Phi_0 + \epsilon E\, \Phi_1(\epsilon) \quad , \; E \neq I$$

However, since we already have $(\Phi_0)_{22} = -(\Phi_0)_{11}'$, premultiplying Φ_0 by an elementary matrix E other than the identity matrix would thus deprive $\hat{\Phi}_0$ of this property. Therefore, $(\hat{\Phi}_\epsilon)_{22} \neq -(\hat{\Phi}_\epsilon)_{11}'$, contradicting our assertion. So we must have $(\Phi_\epsilon)_{22} = (\Phi_\epsilon)_{11}'$. It is easy to see that the optimality conditions of (5.2.21) are the following:

$$\begin{bmatrix} \dot{x}_\epsilon \\ \lambda_{x_\epsilon} \end{bmatrix} = \begin{bmatrix} A_\epsilon - B_\epsilon R_\epsilon^{-1} C_\epsilon & -B_\epsilon R_\epsilon^{-1} B_\epsilon \\ -Q_\epsilon + C_\epsilon R_\epsilon^{-1} C_\epsilon & -(A_\epsilon - B_\epsilon R_\epsilon^{-1} C_\epsilon) \end{bmatrix} \begin{bmatrix} x_\epsilon \\ \lambda_{x_\epsilon} \end{bmatrix}$$

$$\begin{bmatrix} x_\epsilon \\ \lambda_{x_\epsilon} \end{bmatrix}_{t=0} = \begin{bmatrix} x^0 \\ \lambda_{x_\epsilon}^0 \end{bmatrix}$$

$$u_\epsilon^* = -R_\epsilon^{-1} (B_\epsilon \lambda_{x_\epsilon} + C_\epsilon x_\epsilon) .$$

The question of the existence of an optimal problem with optimality conditions and optimal control identical to (5.2.24) and (5.2.26), respectively, is equivalent to the issue of the solvability of $(A_\epsilon, B_\epsilon, C_\epsilon, Q_\epsilon, R_\epsilon)$ in

$$\begin{bmatrix} A_\epsilon - B_\epsilon R_\epsilon^{-1} C_\epsilon & -B_\epsilon R_\epsilon^{-1} B_\epsilon' \\ -Q_\epsilon + C_\epsilon R_\epsilon^{-1} C_\epsilon & -(A_\epsilon - B_\epsilon R_\epsilon^{-1} C_\epsilon)' \end{bmatrix} = \Phi_\epsilon$$

$$-R_\epsilon^{-1} (B_\epsilon' \lambda_{x_\epsilon} + C_\epsilon x_\epsilon) = -R^{-1} (B_2' L_{21} x + (B_1' + B_2' L_{22}) \lambda_x)$$

$$\begin{bmatrix} x_\epsilon \\ \lambda_{x_\epsilon} \end{bmatrix} = \begin{bmatrix} x \\ \lambda_x \end{bmatrix} .$$

Since $(\Phi_\epsilon)_{22} = -(\Phi_\epsilon)_{11}'$, we have a well-posed problem of five equations with five unknowns: $(A_\epsilon, B_\epsilon, C_\epsilon, Q_\epsilon, R_\epsilon)$ viz.

$$A_\epsilon - B_\epsilon R_\epsilon^{-1} C_\epsilon = (\Phi_\epsilon)_{11} \tag{5.2.28}$$

$$-Q_\epsilon + C_\epsilon R_\epsilon^{-1} C_\epsilon = (\Phi_\epsilon)_{21} \tag{5.2.29}$$

$$-B_\epsilon R_\epsilon^{-1} B_\epsilon' = (\Phi_\epsilon)_{12} \tag{5.2.30}$$

$$-R_\epsilon^{-1} C_\epsilon = -R^{-1} B_2' L_{21} \tag{5.2.31}$$

$$-R_\epsilon^{-1} B_\epsilon' = -R^{-1} (B_1' + B_2' L_{22}) \tag{5.2.32}$$

We have shown that the solution $(A_0, B_0, C_0, Q_0, R_0)$ exists for the above equations at $\epsilon = 0$. By the implicit function Theorem, for ϵ small enough, there exists a unique solution of the form

$$B_\epsilon = B_0 + O(\epsilon)$$

$$R_\epsilon = R_0 + O(\epsilon)$$

$$A_\epsilon = A_0 + O(\epsilon)$$

$$C_\epsilon = C_0 + O(\epsilon)$$

$$Q_\epsilon = Q_0 + O(\epsilon)\ .$$

Thus, this completes our proof.

QED

We have shown that a lower order optimal system possesses the same optimality conditions and optimal control as that of the full-order optimal problem with its initial conditions restricted to a manifold. The $(A_\epsilon, B_\epsilon, C_\epsilon, Q_\epsilon, R_\epsilon)$ characterizing the lower order system is of $O(\epsilon)$ perturbation from the $(A_0, B_0, C_0, Q_0, R_0)$ of the reduced optimal problem. The unique existence of $(A_\epsilon, B_\epsilon, C_\epsilon, Q_\epsilon, R_\epsilon)$ is assured by the existence of $(A_0, B_0, C_0, Q_0, R_0)$ and the smallness of the perturbation parameter ϵ. It is worth pointing out that in general a set of optimality conditions does not correspond to a unique optimal problem. As an illustration, consider the optimality conditions

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A - BR^{-1}C & -BR^{-1}B' \\ -Q + CR^{-1}C & -(A - BR^{-1}C)' \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

$$x(0) = x^0 \quad , \quad \lambda(T) = 0 \quad , \quad R > 0$$

It is easy to check that both of the following optimal problems come up with the same optimality conditions as above:

$$\dot{x} = A\,x + B\,u$$

$$\underset{u}{Min}\ J = \underset{u}{Min}\ \frac{1}{2}\int_0^T x'Q\,x + 2u'C\,x + u'R\,u\ dt$$

$$x(0) = x^0 \ ,\ x(T)\ \ free$$

and

$$\dot{x} = (A - BR^{-1}C)x + B\,u$$

$$\min_u J = \min_u \frac{1}{2} \int_0^T x'(Q - CR^{-1}C')x + u'Ru \, dt$$

$$x(0) = x^0 \,, \quad x(T) \text{ free} \,.$$

However, the optimal control in these two optimal problems is different.

5.3. Decomposition of Optimal Linear Systems with Quadratic Criteria

As discussed in the previous Section, a full-order optimal system is equivalent to a lower order one, provided it starts with its initial conditions on the integral manifold. When the initial conditions are not on the manifold, there is a deviation from the manifold. It will be shown in this Section that the optimal trajectory will converge to this slow manifold. This mechanism is analyzed by decoupling the optimality conditions into two. One of these corresponds to the optimal system as restricted to the manifold. The other one governs the behavior of the deviation from the manifold.

We start by looking at the optimality conditions (5.2.7) of the linear-quadratic regulation problem (5.2.1)-(5.2.2):

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda}_x \\ \epsilon \dot{z} \\ \epsilon \dot{\lambda}_z \end{bmatrix} = \begin{bmatrix} A_{11} & -B_1R^{-1}B_1' & A_{12} & -B_1R^{-1}B_2' \\ -Q & -A_{11}' & 0 & -A_{21}' \\ A_{21} & -B_2R^{-1}B_1' & A_{22} & -B_2R^{-1}B_2' \\ 0 & -A_{12}' & -S & -A_{22}' \end{bmatrix} \begin{bmatrix} x \\ \lambda_x \\ z \\ \lambda_z \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} x \\ \lambda_x \\ z \\ \lambda_z \end{bmatrix} \qquad (5.3.1)$$

$$\begin{bmatrix} x \\ \lambda_x \\ z \\ \lambda_z \end{bmatrix}_{t=0} = \begin{bmatrix} x^0 \\ \lambda_x^0 \\ z^0 \\ \lambda_z^0 \end{bmatrix}$$

Define the deviation from the integral manifold $(z, \lambda_z) = L(x, \lambda_x)$ of the closed-loop optimal system (5.3.1) by

$$\begin{bmatrix} \eta \\ \lambda_\eta \end{bmatrix} = \begin{bmatrix} z \\ \lambda_z \end{bmatrix} - L \begin{bmatrix} x \\ \lambda_x \end{bmatrix} . \tag{5.3.2}$$

It is easy to see that with (5.3.2), (5.3.1) is equivalent to the following block triangular system:

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda}_x \\ \epsilon\dot{\eta} \\ \epsilon\dot{\lambda}_\eta \end{bmatrix} = \begin{bmatrix} F_{11} + F_{12} L & F_{12} \\ 0 & F_{22} - \epsilon L\ F_{12} \end{bmatrix} \begin{bmatrix} x \\ \lambda_x \\ \eta \\ \lambda_\eta \end{bmatrix} \tag{5.3.3}$$

$$\begin{bmatrix} x \\ \lambda_x \end{bmatrix}_{t=0} = \begin{bmatrix} x^0 \\ \lambda_x^0 \end{bmatrix} \quad , \quad \begin{bmatrix} \eta \\ \lambda_\eta \end{bmatrix}_{t=0} = \begin{bmatrix} \eta^0 \\ \lambda_\eta^0 \end{bmatrix} = \begin{bmatrix} z^0 \\ \lambda_z^0 \end{bmatrix} - L \begin{bmatrix} x^0 \\ \lambda_x^0 \end{bmatrix} .$$

The optimal control (5.2.6.a) in this new state space is

$$\begin{aligned} u^* &= -R^{-1}(B_1'\lambda_x + B_2'(\lambda_\eta + L_{21}x + L_{22}\lambda_x)) \\ &= -R^{-1}(B_2'L_{21}x + (B_1' + B_2'L_{22})\lambda_x) - R^{-1}B_2'\lambda_\eta = u_M^* + u_\eta^* . \end{aligned} \tag{5.3.4}$$

The optimal control consists of two components, namely u_M^* and u_η^* . The fast control, $u_\eta^* = -R^{-1}B_2'\lambda_\eta$, governs the action of the deviation from the manifold and vanishes as the optimal system is on the integral manifold. Clearly, the optimality conditions (5.3.3) and the optimal control (5.3.4) become identical with (5.2.24)-(5.2.26) when the optimal system starts with its conditions on the manifold or somehow converges to the manifold eventually. We have a decoupled subsystem $(\eta , \lambda_\eta)^T$ from the optimality conditions together with the fast control u_η^*

$$\begin{bmatrix} \epsilon\dot{\eta} \\ \epsilon\dot{\lambda}_\eta \end{bmatrix} = (F_{22} - \epsilon LF_{12}) \begin{bmatrix} \eta \\ \lambda_\eta \end{bmatrix} \quad , \quad \begin{bmatrix} \eta \\ \lambda_\eta \end{bmatrix}_{t=0} = \begin{bmatrix} \eta^0 \\ \lambda_\eta^0 \end{bmatrix} \tag{5.3.5}$$

$$u_\eta^* = -R^{-1}B_2'\lambda_\eta . \tag{5.3.6.a}$$

Note that (5.3.5) is an $O(\epsilon)$ perturbed version of the following optimality condition:

$$\begin{bmatrix} \epsilon\dot{\eta} \\ \epsilon\dot{\lambda}_\eta \end{bmatrix} = F_{22} \begin{bmatrix} \eta \\ \lambda_\eta \end{bmatrix} = \begin{bmatrix} A_{22} & -B_2R^{-1}B_2' \\ -S & -A_{22}' \end{bmatrix} \begin{bmatrix} \eta \\ \lambda_\eta \end{bmatrix} \quad , \quad \begin{bmatrix} \eta \\ \lambda_\eta \end{bmatrix}_{t=0} = \begin{bmatrix} \eta^0 \\ \lambda_\eta^0 \end{bmatrix} . \tag{5.3.6.b}$$

(5.3.6) corresponds to the following optimal problem:

$$\epsilon\dot{\eta} = A_{22}\,\eta + B_2\,u_\eta$$

$$\underset{u_\eta}{Min}\, J_\eta = \underset{u_\eta}{Min}\, \frac{1}{2}\int_0^\infty \eta' S\, \eta + u_\eta'\, R\, u_\eta\, dt$$

$$\eta(0) = \eta^0 \;,\; \eta(\infty) \;\; free \;.$$

For this optimal problem to have a unique optimal solution, we propose the following assumption. A detailed proof can be found on P.237-238 in [32].

Assumption 5.3.1.

$(A_{22}\,,B_2\,,S\,)$ is a stabilizable-detectable triple.

It is easy to see from the block triangular system matrix in (5.3.3) that the decoupled fast subsystem (5.3.5) has 2m fast eigenvalues with order $O(1/\epsilon)$. Among these eigenvalues, m of these eigenvalues are symmetric to the other m eigenvalues with respect to the origin. Recalling the nonsingularity assumption on F_{22} and the smallness of ϵ, it can be similarly shown, as in Theorem 5.2.1, that (5.3.5)-(5.3.6.a) correspond to the following optimal problem:

$$\epsilon\dot{\eta} = A_{22}^{\epsilon}\,\eta + B_2^{\epsilon}\,u_\eta \tag{5.3.7.a}$$

$$\underset{u_\eta}{Min}\, J_\eta = \underset{u_\eta}{Min}\, \frac{1}{2}\int_0^\infty \eta' S_\epsilon\, \eta + u_\eta'\, R_\epsilon\, u_\eta\, dt \tag{5.3.7.b}$$

$$\eta(0) = \eta^0 \;,\; \eta(\infty) \;\; free$$

where

$$(A_{22}^{\epsilon}\;,B_2^{\epsilon}\;,S_\epsilon\,,R_\epsilon) = (A_{22}\,,B_2\,,S\;,R\,) + O(\epsilon)$$

and its unique existence is guaranteed.

When the closed-loop optimal system (5.2.22) starts on the manifold, i. e., $\eta^0 = 0$, we have the following by inspection on (5.3.7):

$$u_\eta^*(t\,) = 0 \qquad ,\; t \geqslant 0\,.$$

With this observation we are again justified that the full-order optimal problem (5.2.1)-(5.2.2) is equivalent to a lower order one, viz. (5.2.21). From standard textbooks on optimal control[31, 32] it is understood that a unique optimal solution to the fast subproblem (5.3.7)

exists provided $(A_{22}^{\epsilon}, B_2^{\epsilon}, S_{\epsilon})$ is stabilizable-detectable. This condition is equivalent to (A_{22}, B_2, S) being stabilizable-detectable by a singular perturbation argument similar to [33]. Due to the presence of ϵ in the state equation (5.3.7.a), the closed-loop fast subsystem tends to zero at the rate of $O(1/\epsilon)$. In other words, this means that the trajectory of the full-order optimal system (5.2.1)-(5.2.2) will converge to the integral manifold at the rate of $O(1/\epsilon)$ and then flow along it slowly as its lower order counterpart described by (5.2.21).

To see how significant the deviation from the manifold contributes to the overall cost, we look at the Hamiltonian-Jacobian Equations of (5.2.1)-(5.2.2) with the standard assumption that $J^*(x, z, t)$ is continuously differentiable on the relevant domain.

$$0 = \frac{1}{2}(x'Qx + z'Sz + u'Ru) + \left(\frac{\partial J^*}{\partial x}\right)'(A_{11}x + A_{12}z + B_1u) + \left(\frac{\partial J^*}{\partial z}\right)'(A_{21}x + A_{22}z + B_2u)\frac{1}{\epsilon}$$

It is pointed out on p. 355 of [31] that the costates $(\lambda_x, \tilde{\lambda}_z)$ are such that

$$\lambda_x = \frac{\partial J^*}{\partial x}, \quad \tilde{\lambda}_z = \frac{\partial J^*}{\partial z}$$

when evaluated along the optimal system $(u^*(t), x^*(t), z^*(t))$.

Consequently,

$$J^* = x'\lambda_x + z'\tilde{\lambda}_z$$

Recall the scaled costate variable $\lambda_z = \dfrac{\tilde{\lambda}_z}{\epsilon}$ that we have been using in the optimality conditions and the optimal controls, the optimal cost can be rewritten as

$$J^* = x'\lambda_x + \epsilon z'\lambda_z .$$

Expressing this cost by the slow state variable (x, λ_x) and the deviation from the integral manifold (η, λ_η), we have

$$\begin{aligned} J^* &= x'\lambda_x + \epsilon(\eta + L_{11}x + L_{12}\lambda_x)'(\lambda_\eta + L_{21}x + L_{22}\lambda_x) \\ &= x'\lambda_x + \epsilon(L_{11}x + L_{12}\lambda_x)'(L_{21}x + L_{22}\lambda_x) + \epsilon((L_{11}x + L_{12}\lambda_x)'\lambda_\eta + \eta'(L_{21}x + L_{22}\lambda_x) + \eta'\lambda_\eta) \\ &= J_M^*(x, \lambda_x) + \epsilon J_\eta^*(x, \lambda_x, \eta, \lambda_\eta). \end{aligned}$$

Note that $J_\eta^* = 0$ for $(\eta, \lambda_\eta)^T = 0$ and it weighs only $O(\epsilon)$ in the overall optimal cost J^*.

We now illustrate the idea of decomposing optimal systems by integral manifold with the following example.

Example 5.3.1

$$\dot{x} = a\,x + b\,z \tag{5.3.8.a}$$

$$\epsilon\dot{z} = -z + u \tag{5.3.8.b}$$

$$\begin{bmatrix} x \\ z \end{bmatrix}_{t=0} = \begin{bmatrix} x^0 \\ z^0 \end{bmatrix}, \qquad \begin{bmatrix} x \\ z \end{bmatrix}_{t=\infty} \; free$$

$$\underset{u}{Min}\, J = \underset{u}{Min}\, \frac{1}{2}\int_0^\infty Qx^2 + Sz^2 + Ru^2\,dt \qquad , \quad R > 0 \;,\; Q \;,\; S \geqslant 0 \tag{5.3.8.c}$$

a, b, Q, S, and R are scalars and $0 < \epsilon << 1$.

Equation (5.3.8) is of an actuator form which is common in practice.

When $\epsilon = 0$ we obtain a reduced optimal problem

$$\dot{x} = a\,x + b\,u \qquad x(0) = x^0 \;,\; x(\infty) = free \tag{5.3.9.a}$$

$$\underset{u}{Min}\, J = \underset{u}{Min}\, \frac{1}{2}\int_0^\infty Qx^2 + (S+R)\,u^2\,dt \tag{5.3.9.b}$$

$$\bar{z} = u \;.$$

Its Hamiltonian is given by

$$H = \frac{1}{2}(Q\,x^2 + (S+R)\,u^2) + \lambda\,(a\,x + b\,u)\,.$$

Necessary conditions for optimality are

$$\frac{\partial H}{\partial u} = b\,\lambda + (S+R)\,u^* = 0$$

whence

$$u^* = \frac{-b}{(S+R)}\,\lambda$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} = -a\,\lambda - Qx \;.$$

Thus,

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} a & -\dfrac{b^2}{S+R} \\ -Q & -a \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}$$

$$x(0) = x^0 \quad , \quad \lambda(T) = 0 .$$

When $\epsilon \neq 0$, the Hamiltonian equation for (5.3.8) is

$$H = \frac{1}{2}(Q\,x^2 + S\,z^2 + R\,u^2) + \lambda_x\,(a\,x + b\,z) + \lambda_z\,(u - z)$$

where $\lambda_z = \dfrac{\tilde{\lambda}_z}{\epsilon}$ is the scaled costate of the original costate variable $\tilde{\lambda}_z$.

Optimality conditions are

$$\frac{\partial H}{\partial u} = \lambda_z + R\,u^* = 0 \qquad , \quad u^* = -\frac{\lambda_z}{R}$$

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = -a\,\lambda_x - Qx$$

$$\dot{\lambda}_z = -\frac{\partial H}{\partial z} = -b\,\lambda_x - Sx + \lambda_z .$$

Hence

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda}_x \\ \epsilon\dot{z} \\ \epsilon\dot{\lambda}_z \end{bmatrix} = \begin{bmatrix} a & 0 & b & 0 \\ -Q & -a & 0 & 0 \\ 0 & 0 & -1 & \dfrac{-1}{R} \\ 0 & -b & -S & 1 \end{bmatrix} \begin{bmatrix} x \\ \lambda_x \\ z \\ \lambda_z \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} x \\ \lambda_x \\ z \\ \lambda_z \end{bmatrix} \tag{5.3.10}$$

$$\begin{bmatrix} x \\ \lambda_x \\ z \\ \lambda_z \end{bmatrix}_{t=0} = \begin{bmatrix} x^0 \\ \lambda_x^0 \\ z^0 \\ \lambda_z^0 \end{bmatrix}$$

$F_{22} = \begin{bmatrix} -1 & -\dfrac{1}{R} \\ -S & 1 \end{bmatrix}$ is nonsingular since $\det(F_{22}) = -(1 + \dfrac{S}{R}) \neq 0$.

Thus, (5.3.10) has an integral manifold

$$\begin{bmatrix} z \\ \lambda_z \end{bmatrix} = L \begin{bmatrix} x \\ \lambda_x \end{bmatrix} = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} x \\ \lambda_x \end{bmatrix}$$

where $L \in R^{2\times 2}$ satisfies

$$F_{21} + F_{22}L = \epsilon L (F_{11} + F_{12}L) .$$

Solving for L,

$$L_{11} = 0$$

$$L_{12} = \frac{\alpha(\epsilon) \pm \beta(\epsilon)}{\gamma(\epsilon)}$$

$$L_{21} = \epsilon RQL_{12}$$

$$L_{22} = -R\ (1 - \epsilon a\ ) L_{12}$$

where

$$\alpha(\epsilon) = -\frac{1}{2\epsilon^2 RQb}$$

$$\beta(\epsilon) = \sqrt{(S + R\ (1 - \epsilon^2 a^2))^2 - 4\epsilon^2 RQb^2}$$

$$\gamma(\epsilon) = \epsilon^2 RQb\ .$$

Thus, on the manifold, (5.3.10) is equivalent to

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda}_x \end{bmatrix} = (F_{11} + F_{12}L(\epsilon)) \begin{bmatrix} x \\ \lambda_x \end{bmatrix} = \begin{bmatrix} a & bL_{12} \\ -Q & -a \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \tag{5.3.11}$$

$$\begin{bmatrix} x \\ \lambda_x \end{bmatrix}_{t=0} = \begin{bmatrix} x^0 \\ \lambda_x^0 \end{bmatrix}$$

$$u^* = -\frac{1}{R}(L_{21}x + L_{22}\lambda_x) = -\epsilon QL_{21}x + (1 - \epsilon a) L_{12}\lambda_x\ .$$

Note that we have two roots for L and we shall pick the one that satisfies

$$L(\epsilon) = L_0 + O(\epsilon) = -F_{22}^{-1}F_{21} + O(\epsilon)$$

where

$$L_0 = -F_{22}^{-1}F_{21} = \begin{bmatrix} -1 & -\frac{1}{R} \\ -S & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 \\ 0 & -b \end{bmatrix} = \begin{bmatrix} 0 & \frac{-b}{S+R} \\ 0 & \frac{bR}{S+R} \end{bmatrix}$$

It is easy to see by applying the L'Hospital's rule that

$$\lim_{\epsilon \to 0} L_{12} = \frac{-b}{S+R}$$

for

$$L_{12} = \frac{\alpha(\epsilon) + \beta(\epsilon)}{\gamma(\epsilon)} .$$

Also,

$$\lim_{\epsilon \to 0} L_{22} = \frac{bR}{S+R} \quad , \quad \lim_{\epsilon \to 0} L_{11} = \lim_{\epsilon \to 0} L_{22} = 0 .$$

With a little effort everyone can see immediately that the scalar optimal problem

$$\dot{x} = a_\epsilon x + b_\epsilon u \tag{5.3.12.a}$$

$$\underset{u}{Min}\, J_\epsilon = \underset{u}{Min}\, \frac{1}{2} \int_0^\infty Q_\epsilon x^2 + 2C_\epsilon xu + R_\epsilon u^2\, dt \tag{5.3.12.b}$$

$$x(0) = x^0 \;,\; x(\infty) \;\; free$$

has the optimality conditions

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda}_x \end{bmatrix} = \begin{bmatrix} a_\epsilon - \frac{b_\epsilon C_\epsilon}{R_\epsilon} & -\frac{b_\epsilon^2}{R_\epsilon} \\ -Q_\epsilon + \frac{C_\epsilon^2}{R_\epsilon} & -(a_\epsilon - \frac{b_\epsilon C_\epsilon}{R_\epsilon}) \end{bmatrix} \begin{bmatrix} x \\ \lambda_x \end{bmatrix}$$

$$\begin{bmatrix} x \\ \lambda_x \end{bmatrix}_{t=0} = \begin{bmatrix} x^0 \\ \lambda_x^0 \end{bmatrix}$$

and optimal control

$$u^* = -\frac{1}{R_\epsilon}(C_\epsilon x + b_\epsilon \lambda_x) .$$

If this optimal problem is to be the lower order equivalent of the full-order optimal system (5.3.8) on the manifold, it is necessary that we have

$$\begin{bmatrix} a & bL_{12} \\ -Q & -a \end{bmatrix} = \begin{bmatrix} a_\epsilon - \dfrac{b_\epsilon C_\epsilon}{R_\epsilon} & -\dfrac{b_\epsilon^2}{R_\epsilon} \\ -Q_\epsilon + \dfrac{C_\epsilon^2}{R_\epsilon} & -(a_\epsilon - \dfrac{b_\epsilon C_\epsilon}{R_\epsilon}) \end{bmatrix} \tag{5.3.13.a}$$

and

$$-\frac{C_\epsilon}{R_\epsilon} = -\epsilon Q L_{12} \tag{5.3.13.b}$$

$$-\frac{b_\epsilon}{R_\epsilon} = (1 - \epsilon a) L_{12} \tag{5.3.13.c}$$

Solving (5.3.13) we have

$$a_\epsilon = a + \frac{\epsilon Q b L_{12}}{(1 - \epsilon a)}$$

$$b_\epsilon = \frac{b}{(1 - \epsilon a)}$$

$$Q_\epsilon = Q - \frac{\epsilon^2 Q^2 b L_{12}}{(1 - \epsilon a)^2}$$

$$R_\epsilon = -\frac{b}{(1 - \epsilon a)^2 L_{12}}$$

$$C_\epsilon = -\frac{\epsilon b Q}{(1 - \epsilon a)^2}$$

Therefore, the optimal control problem (5.3.8) is equivalent to the lower order optimal problem (5.3.12) with the above coefficients. Note that $(a_\epsilon, b_\epsilon, Q_\epsilon, R_\epsilon)$ reduces to $(a, b, Q, (S + R))$ while C_ϵ vanishes at $\epsilon = 0$. This shows that the equivalent lower order optimal problem (5.3.12) becomes the reduced problem (5.3.9) when $\epsilon = 0$. To construct the optimal subproblem that governs the deviation from the manifold, we introduce

$$\begin{bmatrix} \eta \\ \lambda_\eta \end{bmatrix} = \begin{bmatrix} z \\ \lambda_z \end{bmatrix} - L \begin{bmatrix} x \\ \lambda_x \end{bmatrix}$$

as the deviation from the manifold.

With this it is easy to see from (5.3.5)-(5.3.7) that

$$\begin{bmatrix} \epsilon\dot{\eta} \\ \epsilon\dot{\lambda}_\eta \end{bmatrix} = \begin{bmatrix} -1 & -\frac{1}{R} \\ -(S+\epsilon b L_{21}) & 1 \end{bmatrix} \begin{bmatrix} \eta \\ \lambda_\eta \end{bmatrix} \quad , \quad \begin{bmatrix} \eta \\ \lambda_\eta \end{bmatrix}_{t=0} = \begin{bmatrix} \eta^0 \\ \lambda_\eta^0 \end{bmatrix}$$

$$u_\eta^* = -\frac{\lambda_\eta}{R} ,$$

which corresponds to the optimal subproblem

$$\epsilon\dot{\eta} = -\eta + u_\eta \tag{5.3.14.a}$$

$$\underset{u_\eta}{Min}\, J_\eta = \underset{u_\eta}{Min}\, \frac{1}{2}\int_0^\infty (S+\epsilon b L_{21})\,\eta^2 + R\, u_\eta^2 \; dt \tag{5.3.14.b}$$

$$\eta(0) = \eta^0 \ , \ \eta(\infty) \ \ free \ .$$

The stabilizing optimal control gives

$$\epsilon\dot{\eta} = -(1+\frac{K}{R})\eta \quad , \quad \eta(0) = \eta^0$$

where $K > 0$ satisfies the algebraic Riccati equation

$$K - (S+\epsilon b L_{21}) = -K\,(1+\frac{K}{R}) .$$

Thus, $\eta^*(t) \rightarrow 0$ at the rate of $O(1/\epsilon)$. For the case where the in tial conditions $(x^0 , z^0)^T$ do not lie on the manifold, the optimal system will converge to its lower order equivalent at the rate of $O(1/\epsilon)$, and the behavior of the deviation is governed by the optimal subproblem (5.3.14). As a whole, we have shown that

$$\dot{x} = a\,x + b\,z$$

$$\epsilon\dot{z} = -z + u$$

$$\begin{bmatrix} x \\ z \end{bmatrix}_{t=0} = \begin{bmatrix} x^0 \\ z^0 \end{bmatrix} , \qquad \begin{bmatrix} x \\ z \end{bmatrix}_{t=\infty} free$$

$$\underset{u}{Min}\, J = \underset{u}{Min}\, \frac{1}{2}\int_0^\infty Qx^2 + Sz^2 + Ru^2\, dt \qquad , \quad R > 0 \ , \ Q \ , S \geqslant 0$$

is equivalent to the lower order optimal problem

$$\dot{x} = a_\epsilon x + b_\epsilon u$$

$$\underset{u}{Min}\ J_\epsilon = \underset{u}{Min}\ \frac{1}{2}\int_0^\infty Q_\epsilon x^2 + 2C_\epsilon xu + R_\epsilon u^2\, dt$$

$$x(0) = x^0\ ,\ x(\infty)\ free$$

if $(x^0\ ,z^0)^T$ belongs to a manifold.

• **Complete separation into two subproblems**

So far we have seen how a decoupled optimal subproblem concerning the deviation from the manifold is formulated when the initial conditions of the full-order optimal system do not start on the integral manifold. To get a complete separation of the original optimal problem into two subproblems, one slow and one fast, we need to block diagonalize the block triangular matrix in (5.3.3).

Introduce

$$\begin{bmatrix}\xi \\ \lambda_\xi\end{bmatrix} = \begin{bmatrix}x \\ \lambda_x\end{bmatrix} - \epsilon H \begin{bmatrix}\eta \\ \lambda_\eta\end{bmatrix} = \begin{bmatrix}x \\ \lambda_x\end{bmatrix} - \epsilon \begin{bmatrix}H_{11} & H_{12} \\ H_{21} & H_{22}\end{bmatrix}\begin{bmatrix}\eta \\ \lambda_\eta\end{bmatrix}.$$

Differentiate both sides with respect to t

$$\begin{bmatrix}\dot{\xi} \\ \dot{\lambda}_\xi\end{bmatrix} = \begin{bmatrix}\dot{x} \\ \dot{\lambda}_x\end{bmatrix} - \epsilon H \begin{bmatrix}\dot{\eta} \\ \dot{\lambda}_\eta\end{bmatrix}$$

$$= (F_{11} + F_{12}L)\{\begin{bmatrix}\xi \\ \lambda_\xi\end{bmatrix} + \epsilon H \begin{bmatrix}\eta \\ \lambda_\eta\end{bmatrix}\} + F_{12}\begin{bmatrix}\eta \\ \lambda_\eta\end{bmatrix} - H(F_{22} - \epsilon LF_{12})\begin{bmatrix}\eta \\ \lambda_\eta\end{bmatrix}$$

$$= (F_{11} + F_{12}L)\begin{bmatrix}\xi \\ \lambda_\xi\end{bmatrix} + \left[\epsilon(F_{11} + F_{12}L)H + F_{12} - H(F_{22} - \epsilon LF_{12})\right]\begin{bmatrix}\eta \\ \lambda_\eta\end{bmatrix}.$$

Now choose H, which satisfies

$$\epsilon(F_{11} + F_{12}L)H + F_{12} - H(F_{22} - \epsilon LF_{12}) = 0$$

By the implicit function Theorem, a unique solution H $\in R^{2n\ \times 2m}$ to the above equation of the form

$$H = F_{12}F_{22}^{-1} + O(\epsilon)$$

exists for F_{22} being nonsingular and ϵ small enough. With this new state variable we have the

following:

$$\begin{bmatrix} \dot{\xi} \\ \dot{\lambda}_\xi \end{bmatrix} = (F_{11} + F_{12}L) \begin{bmatrix} \xi \\ \lambda_\xi \end{bmatrix} , \quad \begin{bmatrix} \xi \\ \lambda_\xi \end{bmatrix}_{t=0} = \begin{bmatrix} \xi^0 \\ \lambda_\xi^0 \end{bmatrix} \tag{5.3.15}$$

$$\begin{bmatrix} \epsilon\dot{\eta} \\ \epsilon\dot{\lambda}_\eta \end{bmatrix} = (F_{22} - \epsilon LF_{12}) \begin{bmatrix} \eta \\ \lambda_\eta \end{bmatrix} , \quad \begin{bmatrix} \eta \\ \lambda_\eta \end{bmatrix}_{t=0} = \begin{bmatrix} \eta^0 \\ \lambda_\eta^0 \end{bmatrix} \tag{5.3.16}$$

$$u^* = u_s^*(\xi , \lambda_\xi) + u_f^*(\eta , \lambda_\eta)$$

where the optimal control u^* from (5.2.6.a) in this state space consists of two components, namely

$$u_s^*(\xi , \lambda_\xi) = -R^{-1} (B_2' L_{21}\xi + (B_1' + B_2' L_{22})\lambda_\xi) \tag{5.3.17}$$

and

$$u_f^*(\eta , \lambda_\eta) = -R^{-1}B_2' \lambda_\eta \tag{5.3.18}$$

$$+ \epsilon R^{-1} (B_2' L_{21} (H_{11}\eta + H_{12}\lambda_\eta) + (B_1' + B_2' L_{22})(H_{21}\eta + H_{22}\lambda_\eta)) .$$

Comparing (5.3.15) and (5.3.17) with (5.2.24) and (5.2.26), we know at once that the following optimal problem would have optimality conditions and optimal control as (5.3.15) and (5.3.17) respectively.

• **Slow subproblem**

$$\dot{\xi} = A_\epsilon \xi + B_\epsilon u_s \tag{5.3.19.a}$$

$$\underset{u_s}{Min} \, J_s = \underset{u_s}{Min} \, \frac{1}{2} \int_0^\infty \xi' Q_\epsilon \xi + u_s' C_\epsilon \xi + u_s' R_\epsilon u_s \, dt \tag{5.2.21.b}$$

$$\xi(0) = \xi^0 , \quad \xi(\infty) \ free$$

where $(A_\epsilon , B_\epsilon , C_\epsilon , Q_\epsilon , R_\epsilon)$ are as in (5.2.28)-(5.2.32). In the same manner, (5.3.16) is similar to (5.3.5) whereas (5.3.18) is a perturbation of (5.3.6). Thus we have

• **Fast subproblem**

$$\epsilon\dot{\eta} = \tilde{A}_{22}^{\epsilon} \eta + \tilde{B}_2^{\epsilon} u_f \tag{5.3.20.a}$$

$$\underset{u_f}{Min} \, J_f = \underset{u_f}{Min} \, \frac{1}{2} \int_0^\infty \eta' \tilde{S}_\epsilon \eta + u_f' \tilde{R}_\epsilon u_f \, dt \tag{5.3.20.b}$$

$$\eta(0) = \eta^0 \ , \ \eta(\infty) \ free$$

where

$$(\tilde{A}_{22}^{\epsilon} \ , \tilde{B}_2^{\epsilon} \ , \tilde{S}_{\epsilon} \ , \tilde{R}_{\epsilon}) = (A_{22} \ , B_2 \ , S \ , R \ ) + O(\epsilon) \ .$$

Both subproblems exist and are unique.

The transformation

$$\begin{bmatrix} x \\ \lambda_x \\ z \\ \lambda_z \end{bmatrix} = \begin{bmatrix} I & \epsilon H \\ L & I + \epsilon LH \end{bmatrix} \begin{bmatrix} \xi \\ \lambda_\xi \\ \eta \\ \lambda_\eta \end{bmatrix} \tag{5.3.21}$$

is invertible and is expressed by

$$\begin{bmatrix} \xi \\ \lambda_\xi \\ \eta \\ \lambda_\eta \end{bmatrix} = \begin{bmatrix} I + \epsilon HL & -\epsilon H \\ -L & I \end{bmatrix} \begin{bmatrix} x \\ \lambda_x \\ z \\ \lambda_z \end{bmatrix} . \tag{5.3.22}$$

With (5.3.22) we have decomposed the full-order optimal problem (5.2.1)-(5.2.2) into two decoupled optimal subproblems, one slow and one fast. This is an exact decoupling and relies on the nonsingularity assumption of F_{22} and the smallness of ϵ . Instead of solving the $(m + n)th$ order optimal problem (5.2.21)-(5.2.22), we can solve with ease the two lower order subproblems, (5.3.19) and (5.3.20).

- **Optimal problems over a large time interval with prescribed end states**

We now consider the optimal problem of the following singularly perturbed system:

$$\begin{bmatrix} \dot{x} \\ \epsilon\dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \quad x \in R^n \ , z \in R^m \ , u \in R^r \tag{5.3.23}$$

$$\underset{u}{Min} \ J = \underset{u}{Min} \ \frac{1}{2} \int_0^T x'Qx + z'Sz + u'Ru \ dt \tag{5.3.24}$$

$$\begin{bmatrix} x \\ z \end{bmatrix}_{t=0} = \begin{bmatrix} x^0 \\ z^0 \end{bmatrix}, \qquad \begin{bmatrix} x \\ z \end{bmatrix}_{t=T} = \begin{bmatrix} x^T \\ z^T \end{bmatrix},$$

with the same description as in (5.2.1)-(5.2.2).

First of all we note that (5.3.23) is identical to (5.2.1), and (5.2.24) differs from (5.2.2) in that the terminal time is not infinity but T. Also, since the trajectory is required to reach a specified point, the final state conditions are no longer free.

We shall be concerned with T being large and seek an approximate scheme to decompose our optimal problems.

It was shown in a recent paper [34] that an approximate solution to problems of optimal control over a large time interval with end states prescribed

$$\dot{x} = Fx + Gu \qquad x(0),\, x(T) \ \ \textit{prescribed} \tag{5.3.25.a}$$

$$\underset{u}{Min}\; J = \underset{u}{Min}\; \frac{1}{2} \int_0^T x'Qx + u'Ru \; dt \tag{5.3.25.b}$$

$$R > 0,\, Q \geq 0, \text{ and } \{F, \sqrt{Q}\} \textit{ is completely observable}$$

can be obtained by piecing together the optimal trajectory and control of the two infinite time problems

$$\dot{x}^{+} = \frac{dx^{+}}{dt} = Fx^{+} + Gu^{+} \tag{5.3.26.a}$$

$$\underset{u^{+}}{Min}\; J^{+} = \underset{u^{+}}{Min}\; \frac{1}{2} \int_0^\infty x^{+'}Qx^{+} + u^{+'}Ru^{+} \; dt \tag{5.3.26.b}$$

$$x^{+}(0) = x(0) \;,\; x^{+}(\infty) \ \ \textit{free}$$

$$\frac{dx^{-}}{ds} = Fx^{-} + Gu^{-} \tag{5.3.27.a}$$

$$\underset{u^{-}}{Min}\; J^{-} = \underset{u^{-}}{Min}\; \frac{1}{2} \int_0^\infty x^{-'}Qx^{-} + u^{-'}Ru^{-} \; ds \tag{5.3.27.b}$$

$$x^{-}(0) = x(T) \;,\; x^{-}(\infty) \ \ \textit{free}\,.$$

In other words, the solution of a fixed end-point optimal control problem can be approximated by superposition of two regulator problems.

One easy way of piecing is

$$x^*(t) = \begin{cases} x^{-*}(t) & 0 \leqslant t \leqslant t_m \\ x^{\cdot *}(T-t) & t_m < t \leqslant T \end{cases}$$

where t_m is defined to be the time where the two curves meet, ie $x^{-*}(t_m) = x^{\cdot *}(T-t_m)$.

Furthermore,

$$\underset{u}{Min}\, J \sim (\underset{u^-}{Min}\, J^-) + (\underset{u^{\cdot}}{Min}\, J^{\cdot})$$

the sum of the cost of the two regulator problems approaches that of the fixed end-point problem (5.3.25) as $T \to \infty$.

Equation (5.3.27) can be viewed as a regulator system in reverse time, while (5.3.26) is a regulator system in forward time.

Applying this scheme to our fixed end-point, linear-quadratic regulator problem (5.3.23)-(5.3.24), we have the forward regulator formulated as

$$\begin{bmatrix} \dot{x}^- \\ \epsilon \dot{z}^- \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^- \\ z^- \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u^- \quad x^- \in R^n \;, z^- \in R^m \;, u^- \in R^r$$

$$\underset{u^-}{Min}\, J^- = \underset{u^-}{Min}\, \frac{1}{2} \int_0^\infty x^{-'} Q x^- + z^{-'} S z^- + u^{-'} R u^- \, dt \tag{5.3.28}$$

$$\begin{bmatrix} x^- \\ z^- \end{bmatrix}_{t=0} = \begin{bmatrix} x^0 \\ z^0 \end{bmatrix}, \qquad \begin{bmatrix} x^- \\ z^- \end{bmatrix}_{t=T} \quad free$$

which is equivalent to the following lower order optimal problem when the initial conditions belong to a manifold.

$$\dot{x}_\epsilon^- = A_\epsilon^- x_\epsilon^- + B_\epsilon^- u_\epsilon^-$$

$$\underset{u_\epsilon^-}{Min}\, J_\epsilon^- = \underset{u_\epsilon^-}{Min}\, \frac{1}{2} \int_0^\infty x_\epsilon^{-'} Q_\epsilon^- x_\epsilon^- + u_\epsilon^{-'} C_\epsilon^- x_\epsilon^- + u_\epsilon^{-'} R_\epsilon^- u_\epsilon^- \, dt$$

$$x_\epsilon^-(0) = x^0 \;, \; x_\epsilon^-(\infty) \;\; free$$

and the deviation from the manifold is an independent optimal problem described by

$$\epsilon\dot{\eta}^- = A_{22}^{\epsilon^-}\eta^- + B_2^{\epsilon^-} u_\eta^-$$

$$\underset{u_\eta^-}{Min} J_\eta^- = \underset{u_\eta^-}{Min} \frac{1}{2}\int_0^\infty \eta^{-\prime} S_\epsilon^- \eta^- + u_\eta^{-\prime} R_\epsilon^- u_\eta^- \, dt \tag{5.3.29}$$

$$\eta^-(0) = \eta^{0-} \ , \ \eta^-(\infty) \ free \ .$$

The reverse regulator problem can be similarly defined and decomposed.

Overall, we propose the following approximate optimal control solution to the fixed end-point optimal problem (5.3.25) by composing the solutions of several lower optimal problems as

$$u^*(t) = u^{*-}(t) + u^{*+}(T-t) + u^*_{\eta^+}(t) + u^*_{\eta^-}(T-t)$$

By the above analysis, we can picture the optimal trajectory of the fixed end-point optimal problem as shown in Figure 5.1. The trajectory converges to a manifold asymptotically and then flow along it until it approaches the vicinity of the terminal time, where it leaves the manifold and goes to the designated end states. It is $u^*_{\eta^+}$ that brings the optimal trajectory to the manifold. On the other hand, $u^*_{\eta^-}$ is responsible for steering the trajectory away from the manifold and going to the prescribed final points.

6. CONCLUSIONS

We have presented the necessary and sufficient conditions for the existence of the integral manifold in linear systems. It has been shown that in linear systems there exists a family of input dependent integral manifolds if the existence of the manifold is assured in the zero input case. The relationship between them is also given. Emphasis is given, though not exclusively, to singularly perturbed systems. A two-stage design in the eigenvalue placement problem clearly illustrates the usefulness of the integral manifold approach in reducing the computational complexity and a way to obtain an approximate solution in the singularly perturbed systems. The effect of parasitics on the nominal states of the singularly perturbed systems can be taken into account by using the manifold idea. This allows us to design a controller to achieve the tracking objective to any order of accuracy in such systems. All of these results are described in Chapter 2.

In Chapter 3 we applied our results to the control problem of flexible link manipulators. The nonlinear system in this case was shown to be a perturbed version of a linear time-invariant system. The unsatisfactory performance of flexible robots at high-frequency maneuvers is explained by the fact that the presence of neglected flexibility causes a phase delay in the system output. Time domain analysis using the integral manifold approach provides a corrective scheme which coincides with that based on the frequency domain analysis. However, in a more general model where the Laplace transform is not applicable due to nonlinearity or time-varying characteristics in the system, the integral manifold approach is still applicable and offers a solution in controller design. To extend our idea to the flexible joint manipulators, we studied the effect of a flexible connection in an interconnected mechanical system which also includes the flexible joint robot model as one of its kind. The flexible system has a *displaced center of mass* with respect to that of the rigidly connected one. The flexibility also induces a perturbed natural frequency and a perturbed damping ratio.

We focused on the problem of tracking and disturbance rejection in the nonlinear system in Chapter 4. For a class of *linear equivalent nonlinear system*, linearizing and stabilizing the nonlinear system render an integral manifold globally attractive. Based on this observation, we designed a slowly varying integral control that drives the system along the manifold to its equilibrium where the tracking error becomes zero. Regardless of initial conditions, asymptotic tracking and disturbance rejection of slowly varying signals can be achieved due to the global attractivity of the manifold.

It is known[35] that the optimal trajectory of a long-range flight consists of steep ascending to a manifold, cruising along it with a fairly constant altitude, and finally descending to the destination. This is precisely a typical optimal trajectory of a singularly perturbed system. This leads to the analysis in Chapter 5. When the initial conditions are restricted to a manifold, the optimal problem of a singularly perturbed linear system with quadratic cost functional is shown to be equivalent to a lower order one. Regulation problems as well as optimal problems over a large time interval with prescribed end states are both studied.

The tracking problem for flexible link robots can be viewed as controlling the robot trajectory to a prescribed attractive manifold. This is certainly one of the future research areas. Among other prominent research issues, the minimum-time-to-climb problem in aerodynamics falls into the category of the singularly perturbed optimal system with constraints. An extension of ideas in Chapter 5 should be done to solve this well-known problem.

We have shown the use of the integral manifold in system designs through the limited scope of this thesis. Extension of these ideas to other areas of system and control will definitely be a future work with a unified theory and with countless applications.

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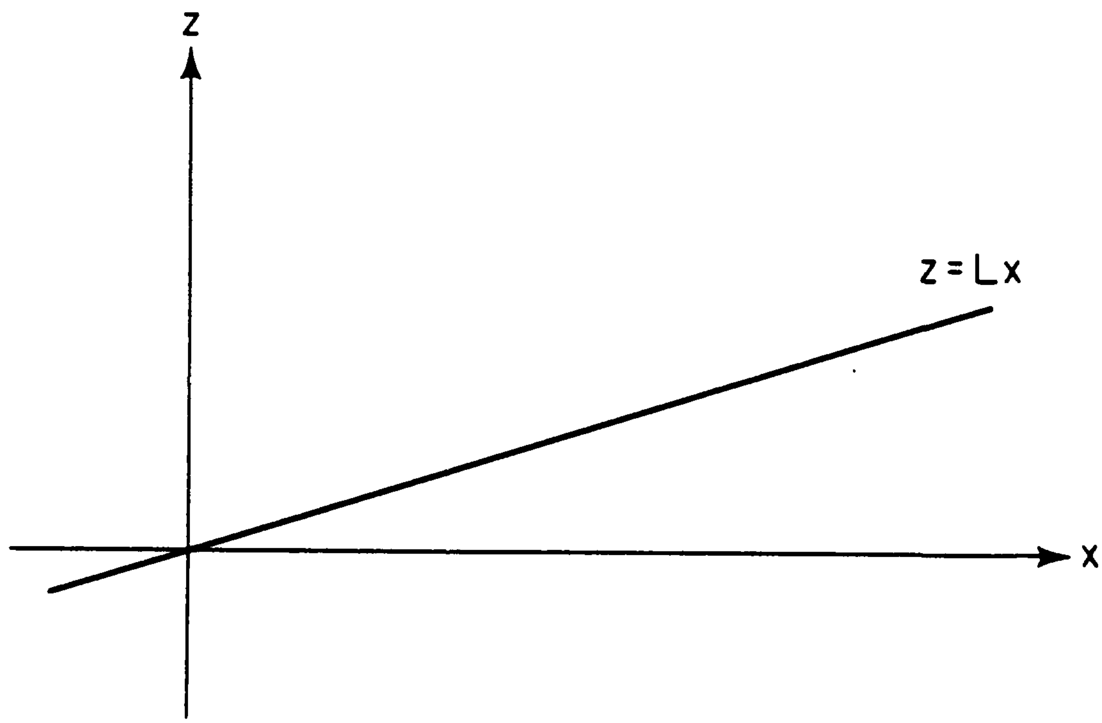

Figure 2-1 Integral manifold $z = Lx$ with both z and x being scalars.

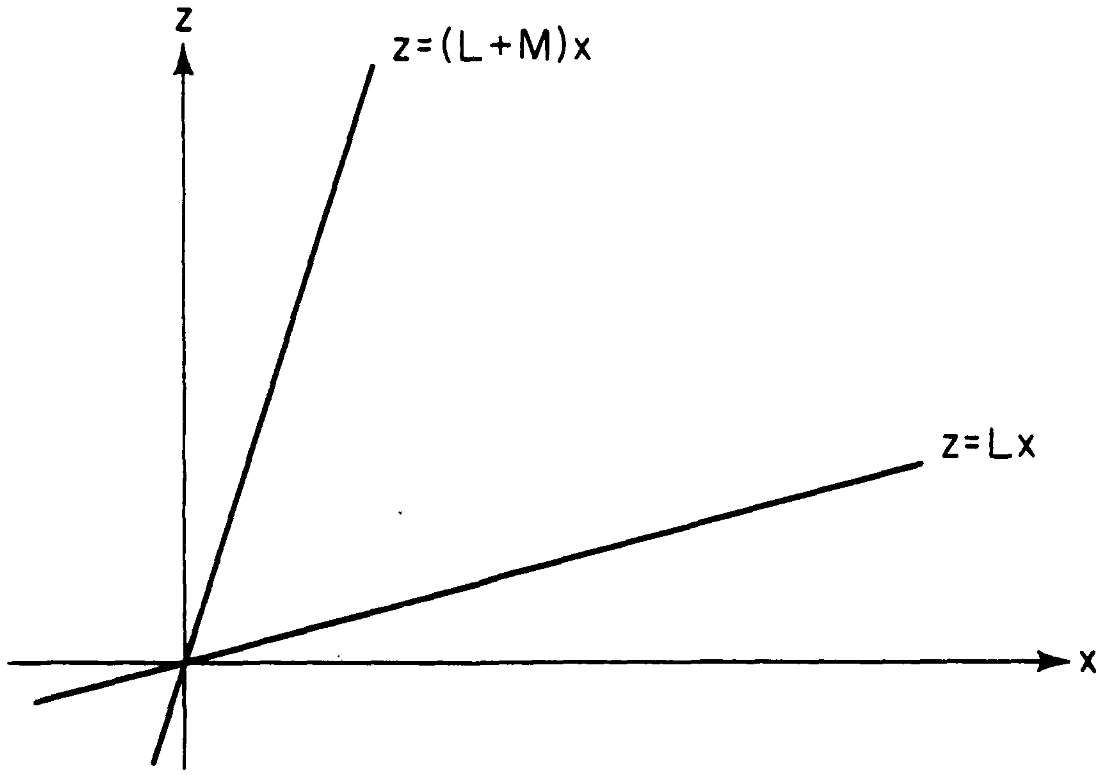

Figure 2-2 The closed-loop system has a new shifted manifold.

$$\begin{bmatrix}\dot{x}\\ \epsilon\dot{z}\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}x\\ z\end{bmatrix} + \begin{bmatrix}B_1\\ B_2\end{bmatrix}u$$

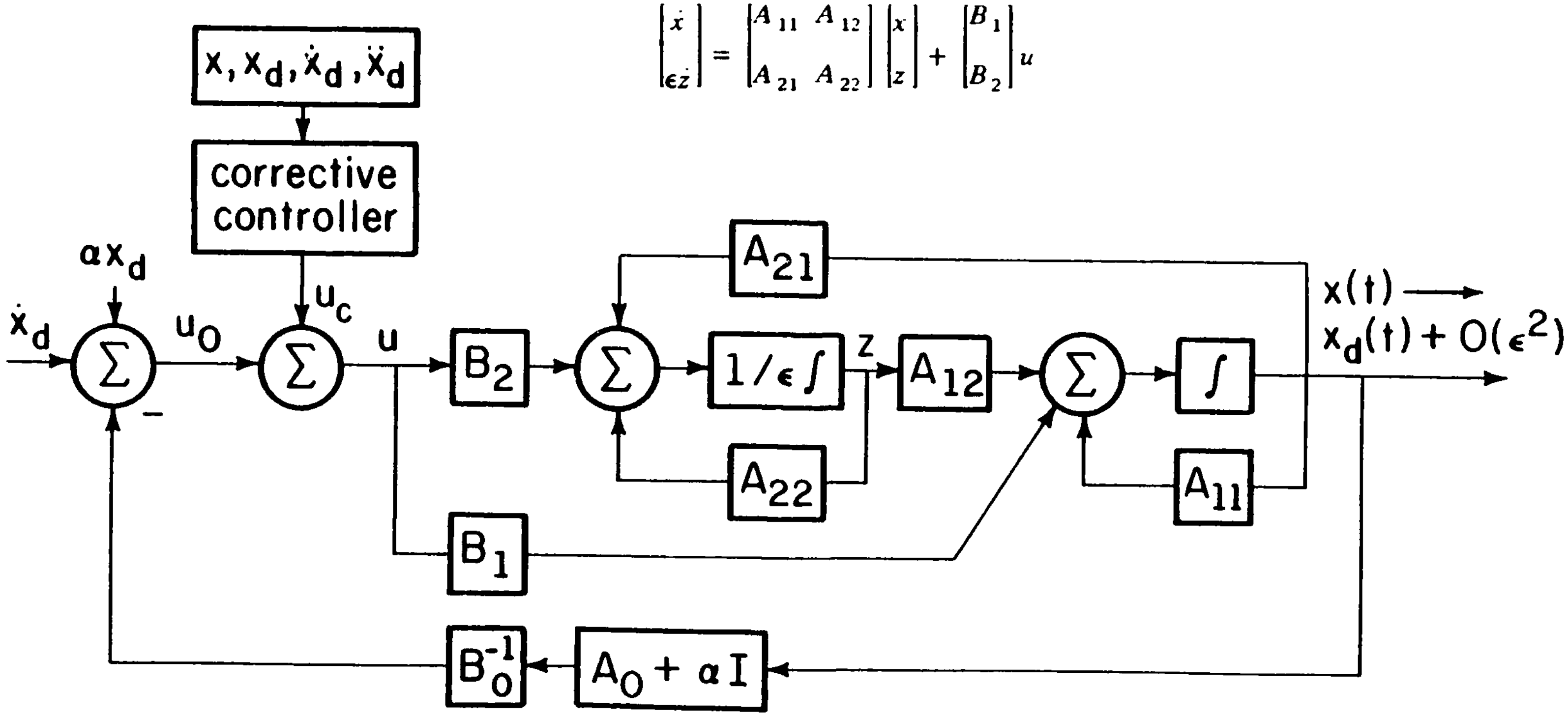

Figure 2-3 Controller with corrective feedback

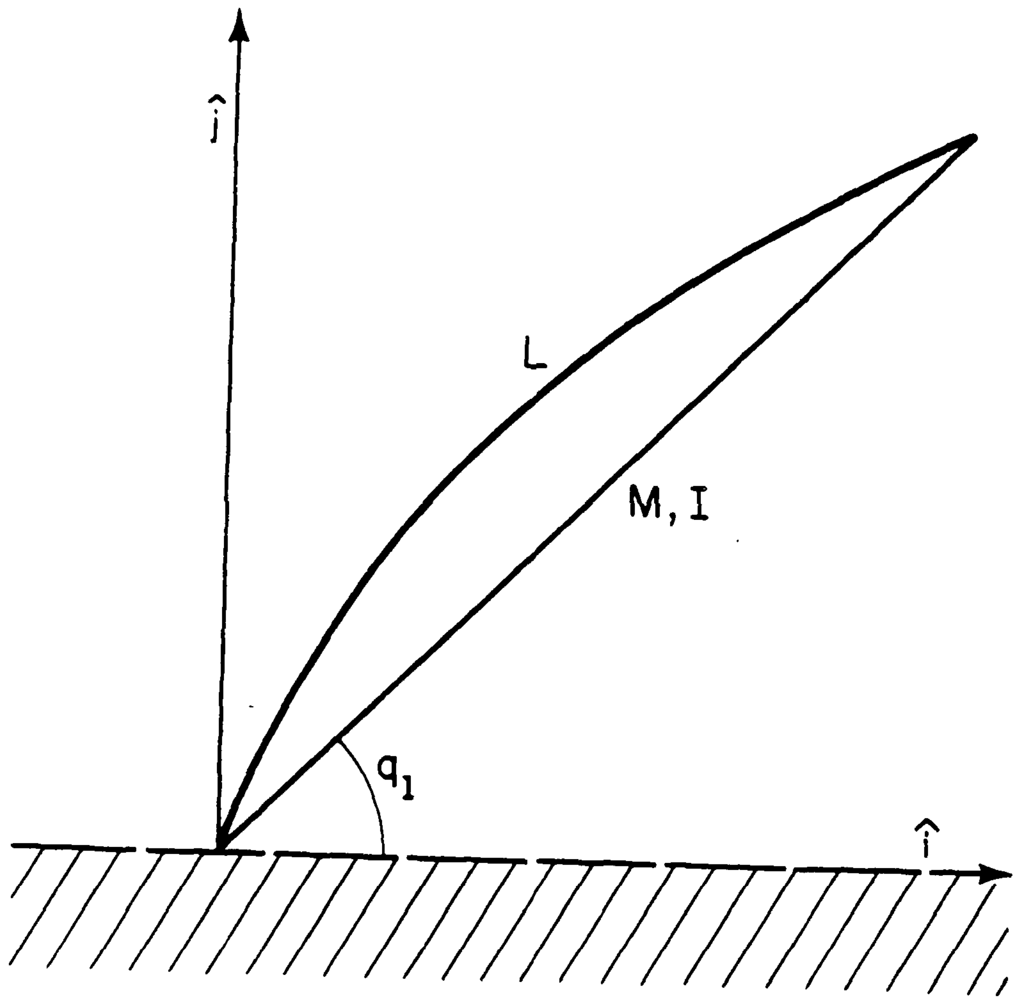

Figure 3-1 Planar flexible beam.

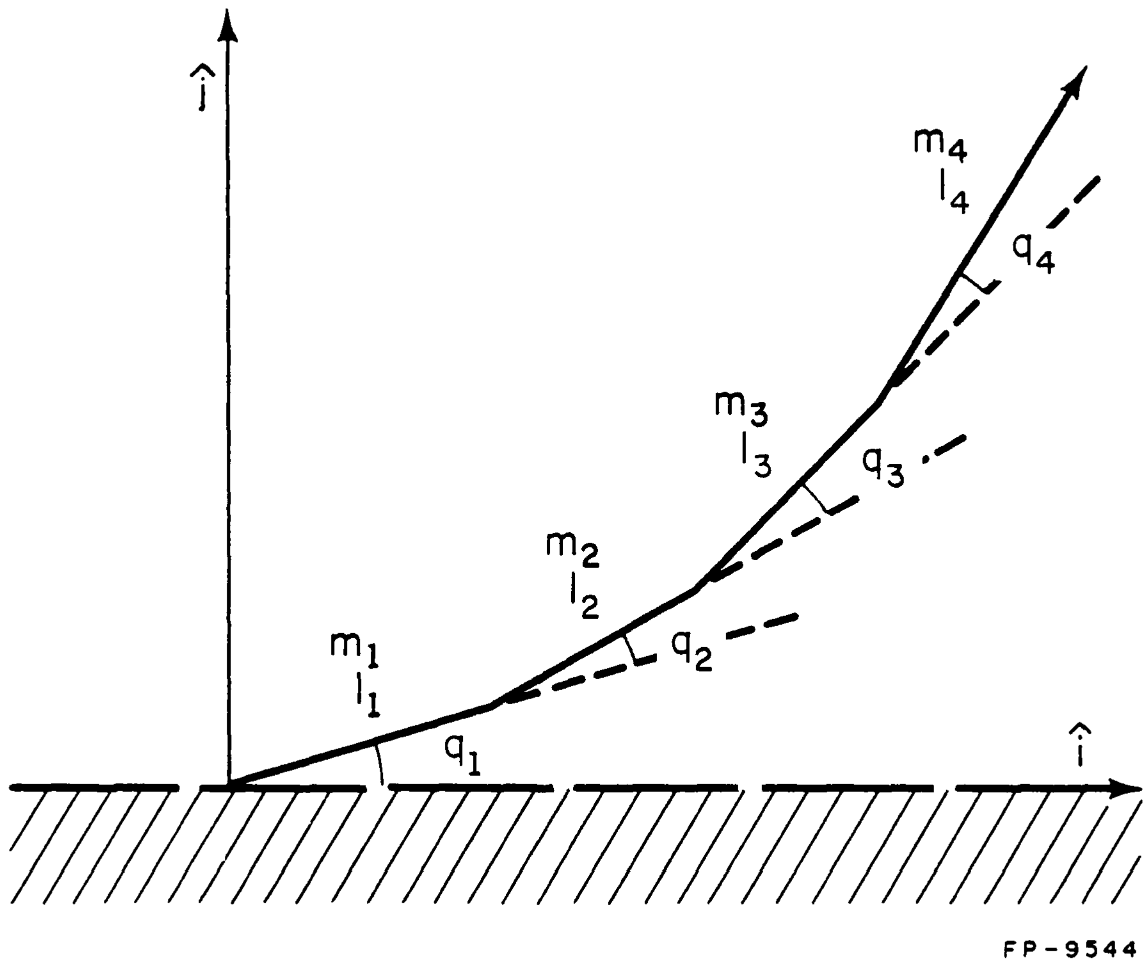

Figure 3-2 Flexible beam modeled as n rigid interconnected sublinks.

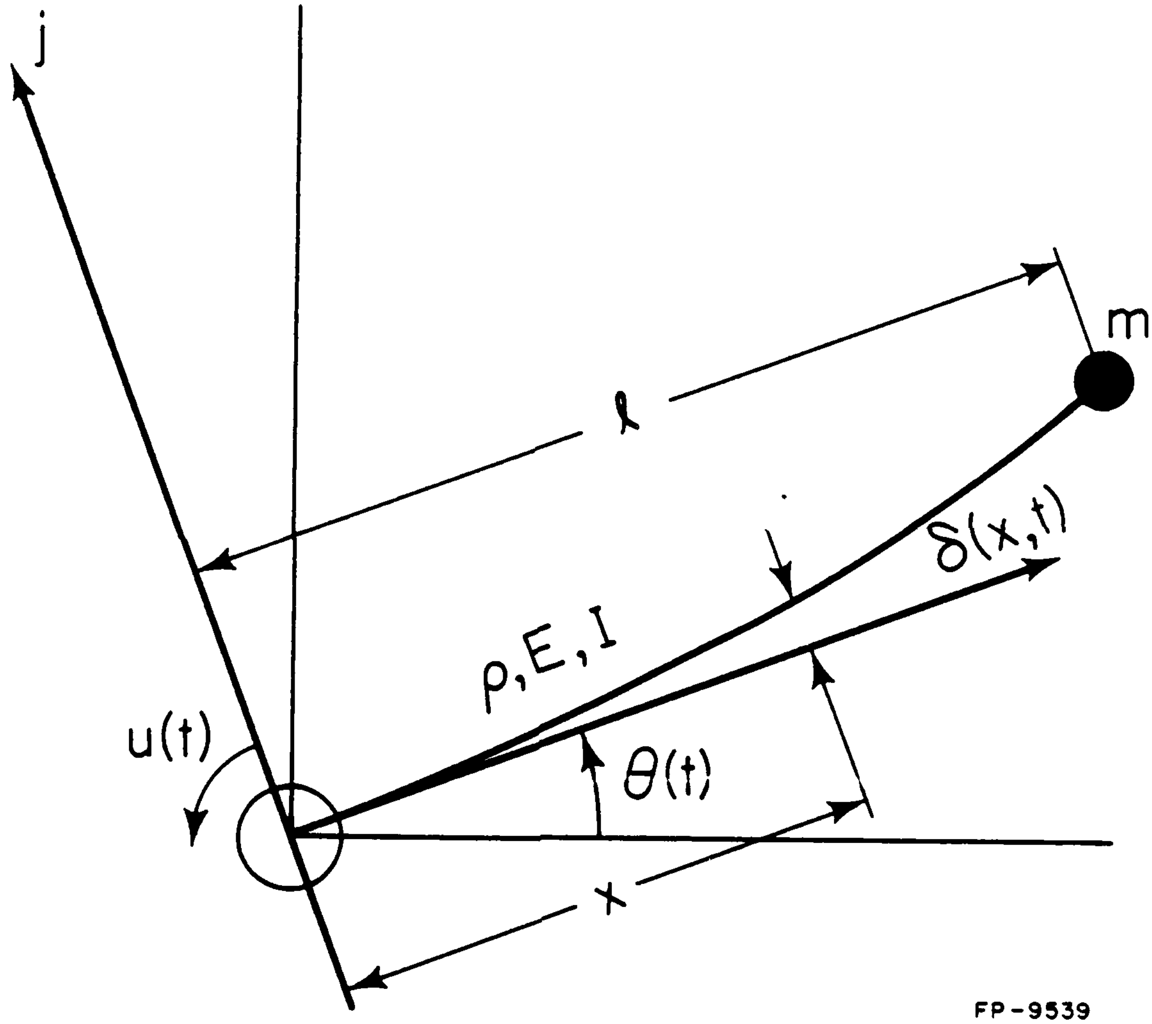

Figure 3-3 Flexible beam with mass m at its tip.

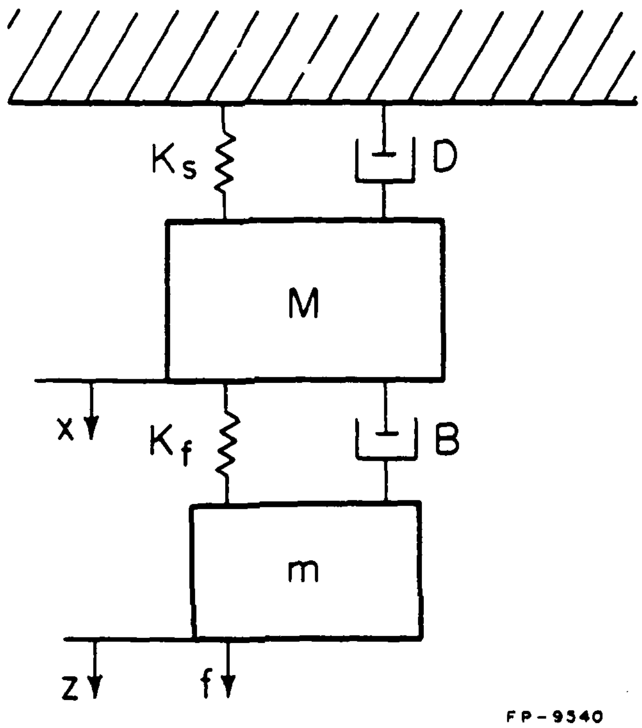

Figure 3-4 Interconnected mechanical system.

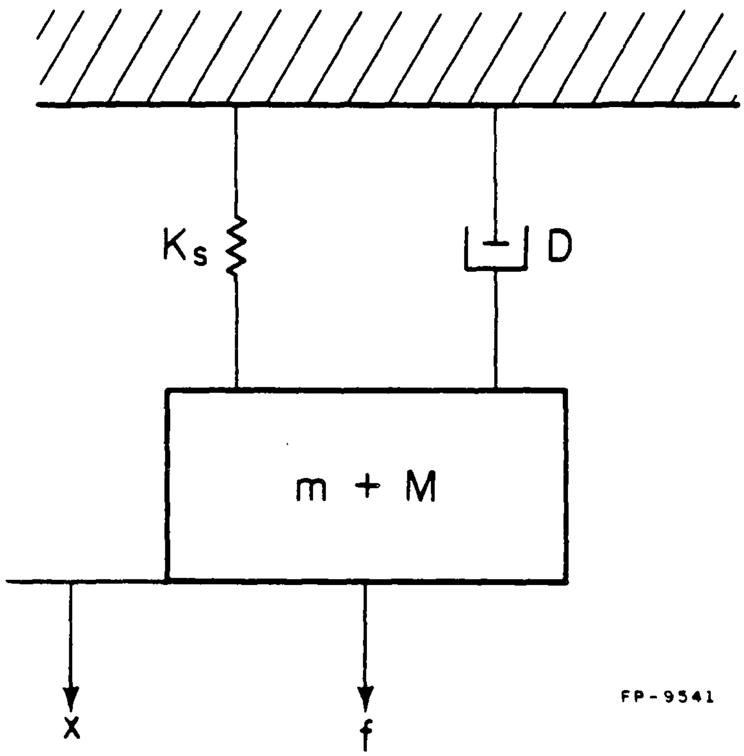

Figure 3-5 Mechanical system with rigid connection.

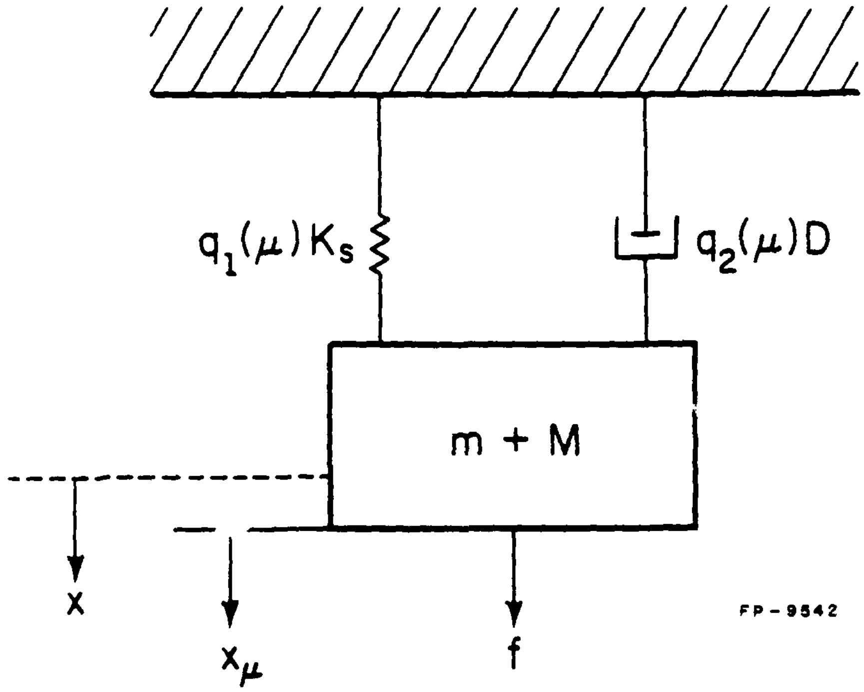

Figure 3-6 Interconnected mechanical system as a whole.

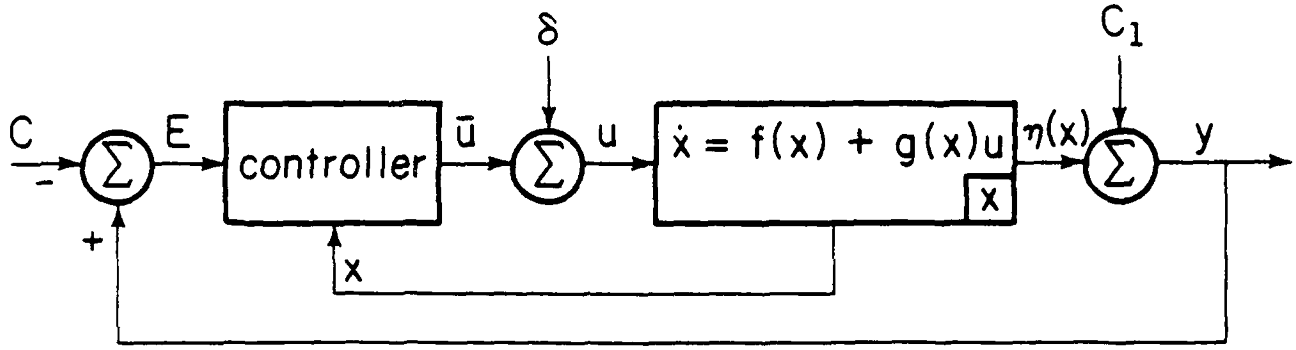

Figure 4-1 Nonlinear system with nonlinear output.

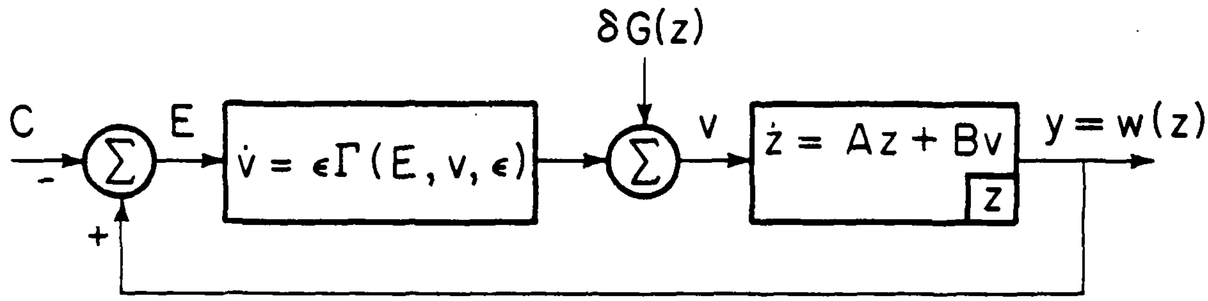

Figure 4-2 External feedback linearized system with nonlinear output.

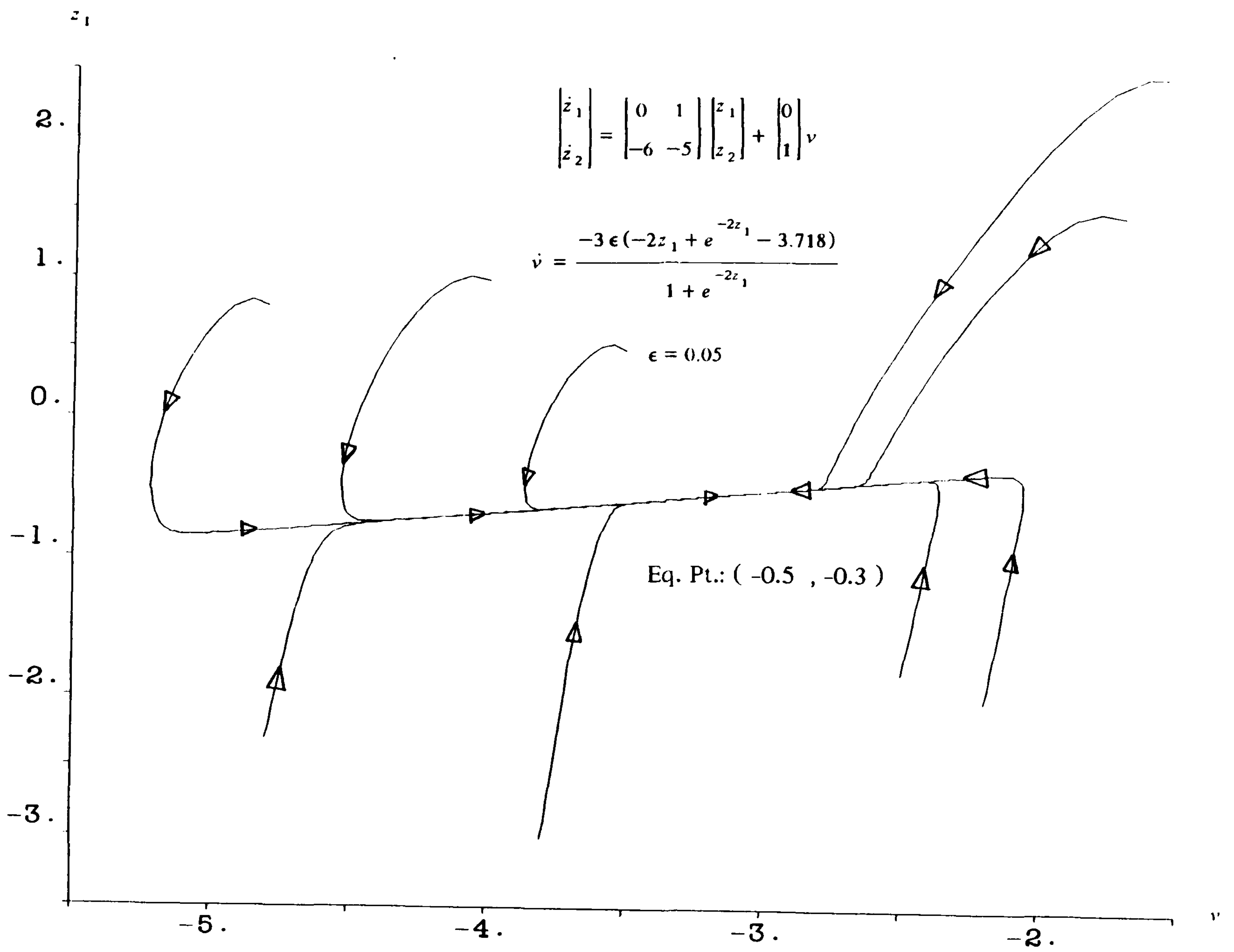

Figure 4-3 System trajectories converge to a manifold where perfect tracking is achieved.

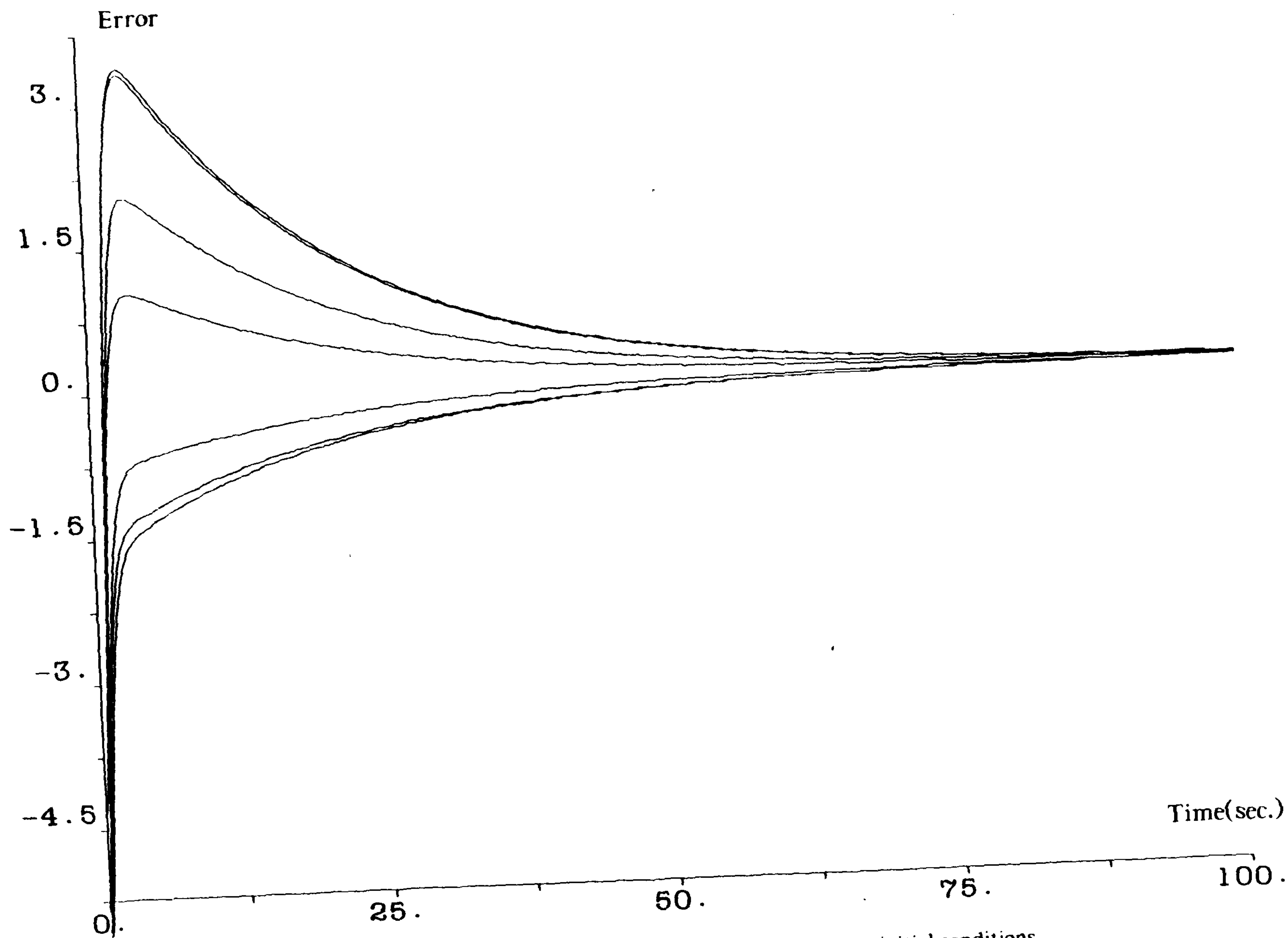

Figure 4-4 Simulations of asymptotic tracking with different initial conditions.

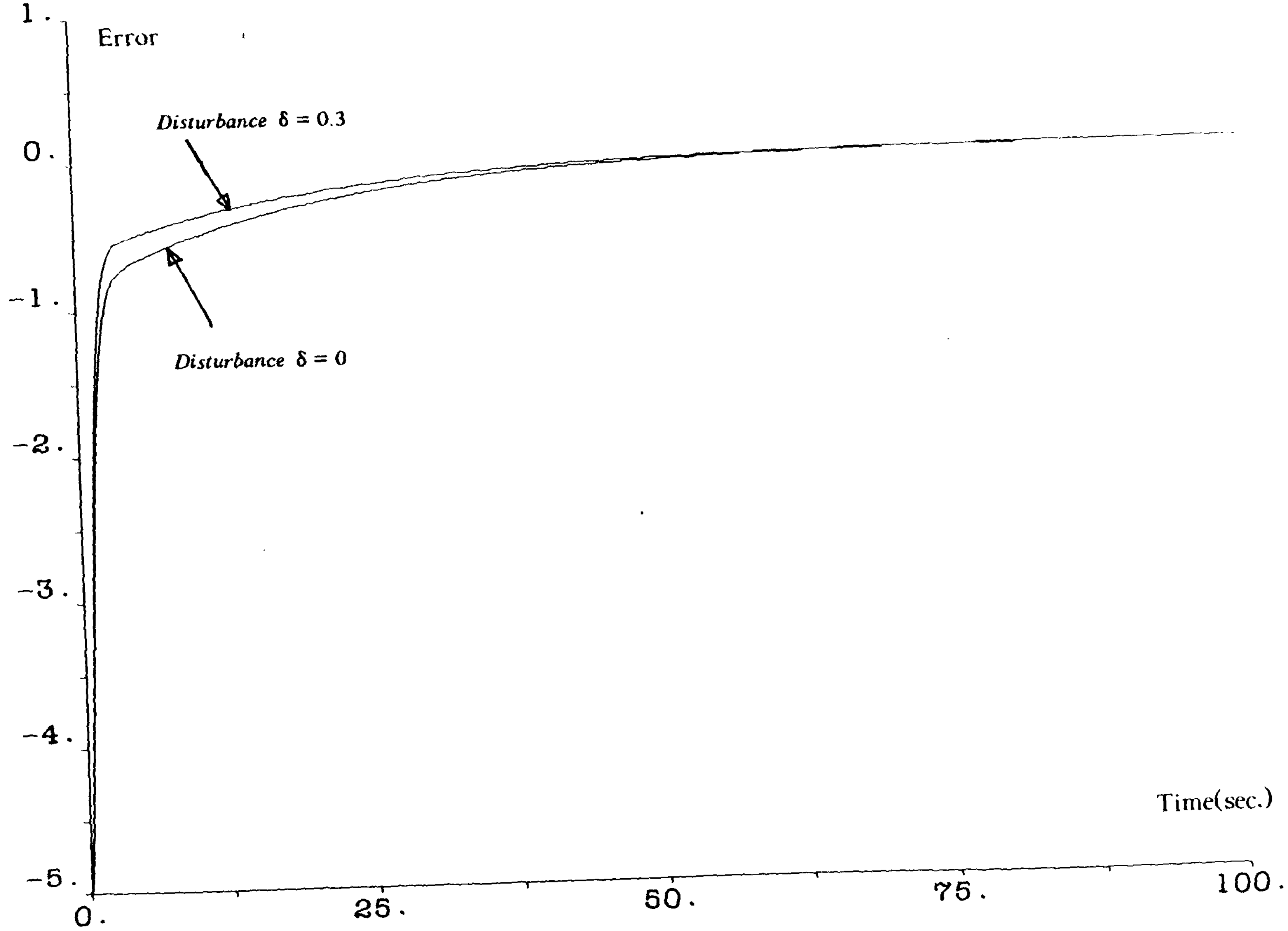

Figure 4-5 Asymptotic tracking and disturbance rejection of constant bounded signals.

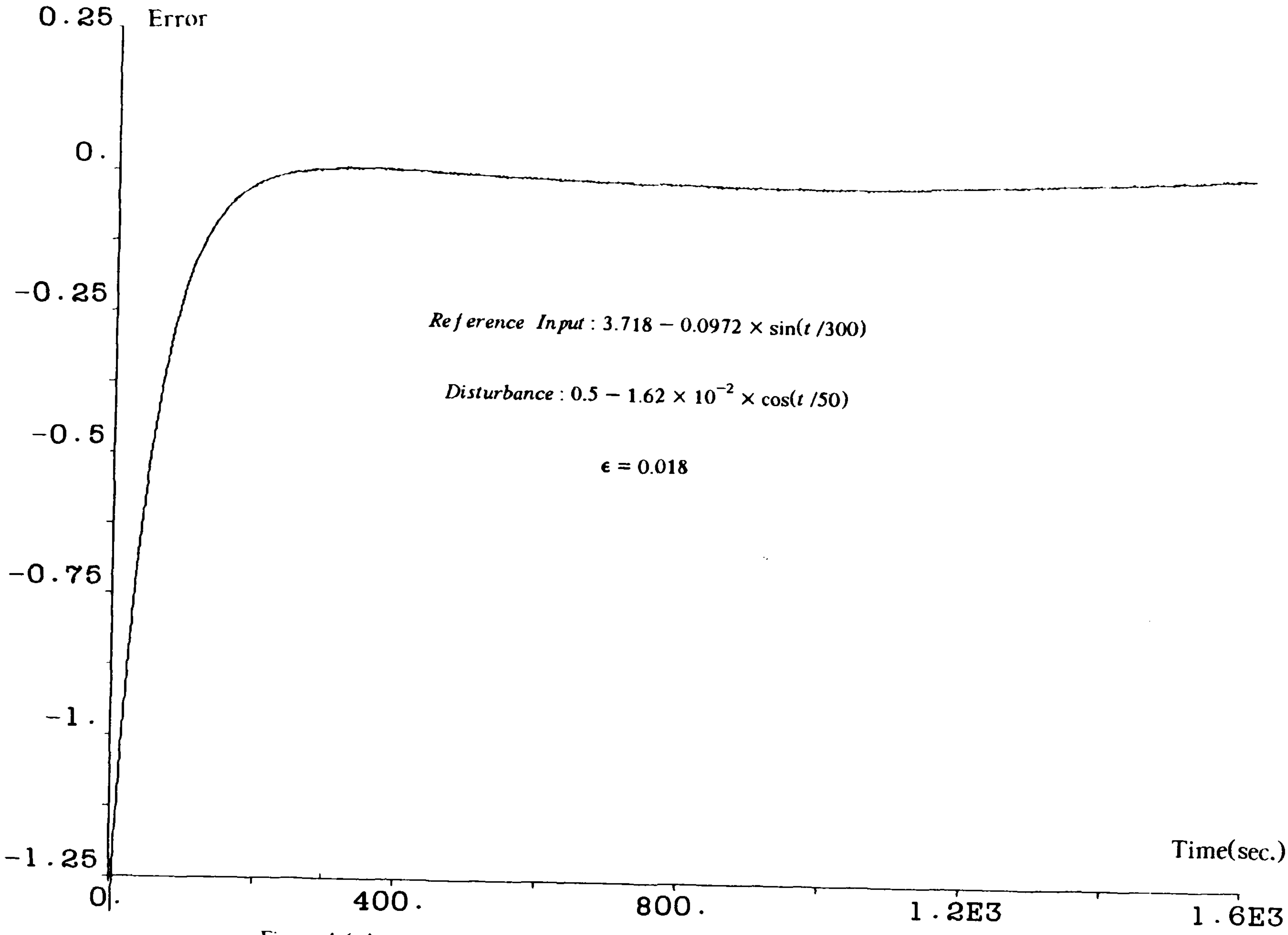

Figure 4-6 Asymptotic tracking and disturbance rejection of slowly varying bounded signals.

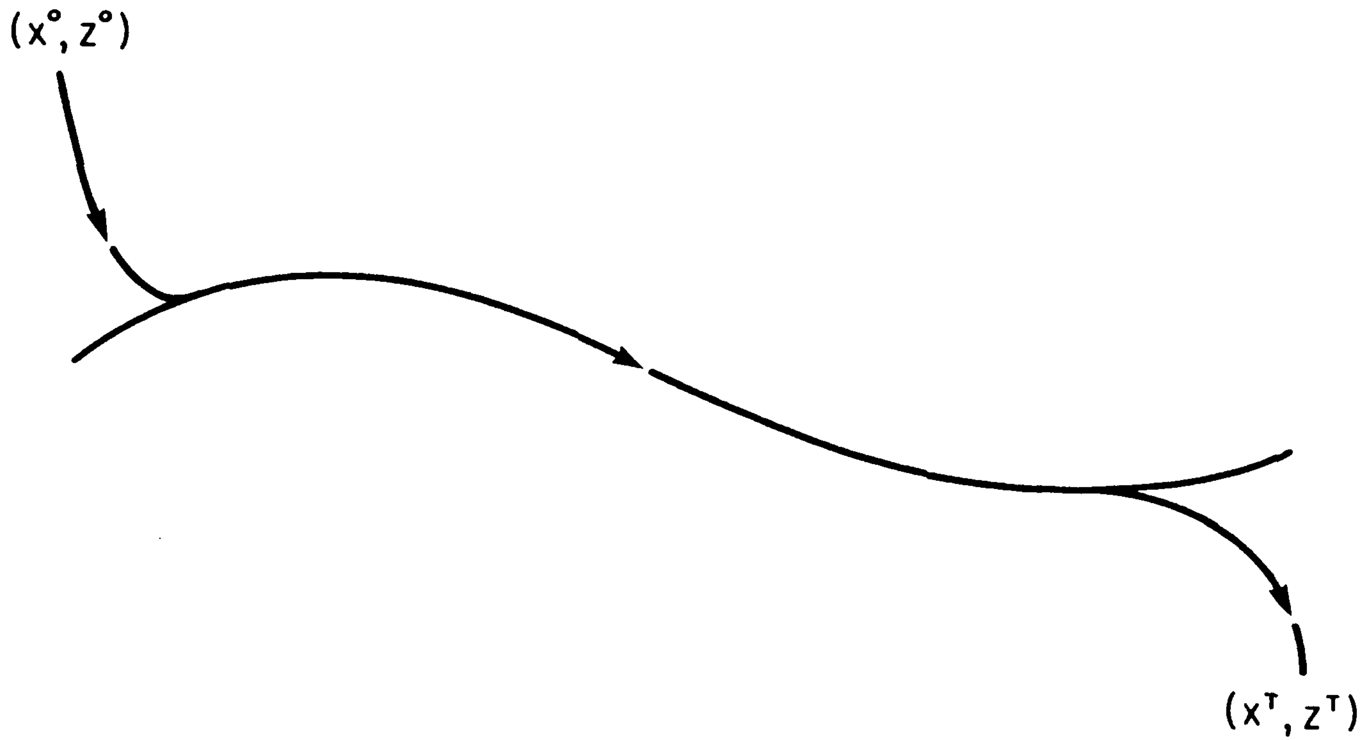

Figure 5-1 Optimal trajectory of a fixed end-point optimal problem.

VITA

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